

## A quick definition of the relativistic expression of momentum

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A detailed analysis of how to introduce the relativistic expression of momentum is made in Luca Peliti's paper [1]. We give here an argument that, on the contrary, appeals to the solution of a functional equation. Newton's second law requires that the variation with time of the momentum is the same in any inertial reference frame according to the principles of special relativity. The search for any velocity-dependent property of a particle whose variation with time is independent of the inertial reference frame leads to a very simple functional equation. The resolution of this functional equation leads us to the relativistic expression of momentum. Consider two inertial frames R and R' which are moving in a uniform translational motion in the direction of the x-axis. A particle moves along this x-axis. Let V be the velocity of R' with respect to R, u the velocity of the particle with respect to R and u' with respect to R'.

Let us recall the formulas resulting from the Lorentz transformation:

$$dt' = \gamma(V) \left(1 - \frac{uV}{c^2}\right) dt ; \quad u' = \frac{u-V}{1-\frac{uV}{c^2}} ; \quad \text{where } \gamma(V) = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \quad (1)$$

Consider a velocity-dependent property of the particle: G(u) or G(u'). We look for this function so that its temporal variation is the same in all inertial frames of reference, i.e.:

$$\frac{dG(u)}{dt} = \frac{dG(u')}{dt'} \quad (2)$$

By defining:  $g(u) = \frac{dG(u)}{du}$  and  $g(u') = \frac{dG(u')}{du'}$  and using equations (1), (2) transforms into:

$$g(u) \frac{du}{dt} = g(u') \frac{du'}{dt'} = g(u') \frac{du'}{du} \frac{du}{dt} = g \left( \frac{u-V}{1-\frac{uV}{c^2}} \right) \frac{1}{[\gamma(V)(1-\frac{uV}{c^2})]^3} \frac{du}{dt} \quad (3)$$

If we eliminate the trivial case where the acceleration  $\frac{du}{dt}$  is zero, we get:

$$g(u) = g \left( \frac{u-V}{1-\frac{uV}{c^2}} \right) \frac{1}{[\gamma(V)(1-\frac{uV}{c^2})]^3} \quad (4)$$

It is a functional equation where  $g(u)$  is the unknown function with  $-c < u < c$ . This equation must, in particular, be verified for  $u = V$  (the same result is obtained by making  $u = 0$ ):

$$g(V) = g(0) \frac{1}{[\gamma(V)]^3 (1-\frac{V^2}{c^2})^3} = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{3}{2}} \text{ with } -c < V < c .$$

Thus,  $\xi$  being a variable between -c and c:

$$g(\xi) = g(0) \left(1 - \frac{\xi^2}{c^2}\right)^{-\frac{3}{2}} \quad (5)$$

We can then verify that  $g(\xi)$  satisfies equation (4)

The function G(u) is obtained by integrating (5), assuming that this function cancels for  $u = 0$  and calling  $m_0$  the constant  $g(0)$  :

$$G(u) = \int_0^u g(0) \left(1 - \frac{\xi^2}{c^2}\right)^{-\frac{3}{2}} d\xi = \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} \quad (6)$$

Here we recognize the relativistic momentum. In such a way we also demonstrated that momentum  $G(u)$  is the only velocity-dependent property of the particle which non-zero temporal variation is the same in all inertial frames of reference.

[1] Luca Peliti; Elementary derivation of the expressions of momentum and energy in special relativity; arxiv:1504.07287v2; 6 feb 2016.