Supplementary Material for 'Exceptional points of PT-symmetric reflectionless states in complex scattering systems'

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I. EXPERIMENTAL SETUPS

A. Multi-Mirror Fabry-Perot Interferometer

In Fig. 1(b,c) of the main text, we have presented a colormap of the reflection $R(\nu, z) = |r(\nu, z)|^2$ with respect to the frequency ν and the depth z of the two rods inserted within the waveguide when a single dielectric alumina slab was placed between each coax-to-waveguide transition and the waveguide. The aim of this slab is to enhance the reflectivity at the interface and the Q-factor of the resonances. The collapse of RL states into RL-EPs however does not depend on this reflectivity and is a more general phenomenon. In Fig. S1, we show the same colormap for 0 and 2 dielectric slabs at each interface.

Remarkably, the system still acts as a Fabry-Perot interferometer even without the slabs with regularly spaced RL-frequencies in absence of the perturbation. The reflectivity is very small but non-vanishing as the leads (coax-to-waveguide transitions) are well but not perflectly matched to the waveguide. The reflection average over the frequency range and the Q-factor of the resonances are small leading to broad zeros in reflection. In contrast, for 2 dielectric slabs, the resonances with large Q-factors are peaked and the EPs are narrower. In both cases, As z increases, RL-EPs can be oberseved as z increases.

We estimate the reflectively R_0 at the interfaces pro-



FIG. S1. Colorscale representation of the reflection on a dB-scale, $10\log[R(\nu, z)]$, measured with a Vector Network Analyzer between 8 and 10 GHz with zero (a) and two (b) alumina slabs at each interface.



FIG. S2. Spectrum of the numerical (blue) and experimental (red) reflection for an empty waveguide with one alumina slab at each interface.

vided by 0, 1 or 2 dielectric slabs by fitting the spectrum of the reflection $R(\nu, z = 0)$ for an empty waveguide with its theoretical formula for a Fabry-Perot cavity

$$R = 1 - \frac{(1 - R_0)^2}{1 + R_0^2 + 2R_0 \cos \Delta \phi},$$
 (S1)

with $\Delta \phi = 2kL$ quantifying the phase shift of the wave propagating from one Fabry-Perot mirror to the other. Here $k = \frac{2\pi}{\lambda}$ is the wave number and L the distance between the mirrors. The spectrum of the reflection and its best fit for a single dielectric slab placed at each interface is shown in Fig. S2. We estimate that $R_0 = 0.002$, $R_0 = 0.25$ and $R_0 = 0.46$ for 0, 1 and 2 slabs, respectively.

B. Disordered one-dimensional medium

As mentioned in the main text, RL-EPs can exist in any complex multiple-scattering systems. We add a random single-channel disorder made of the same 28 metallic spheres as those presented in the main text inside the waveguide. The reflection $R(\nu, z)$ is shown in Fig. S3. Without perturbation (z = 0), the spectrum presents randomly spaced dips. In contrast to the Fig. 3(a), one can see less RL-EPs with the random disorder than the symmetric one. Indeed, with a symmetrical disorder, the RL-eigenvalues can leave the real axis only by coalescing with another RL-eigenvalue whereas with a random disorder, a RL-eigenvalue can randomly leave the real axis by itself.





FIG. S3. Colorscale representation of $R(\nu, z)$ for a random disordered medium without mirror symmetry.

II. THEORETICAL ANALYSIS

In this section, we provide more details on the calculations using the effective two-level Hamiltonian approach. The effective Hamiltonian writes

$$H_{\rm eff} = H_0 - \frac{i}{2} V_0 V_0^T - \frac{i}{2} V_1 V_1^T, \qquad (S2)$$

where the left and right vectors V_0 and V_1 account for the coupling of the channels to the closed system. In the basis of two modes of the closed waveguide, H_0 is a diagonal matrix with elements $\omega_0 - \delta\omega_0/2$ and $\omega_0 + \delta\omega_0/2$. We have identified that the coupling vectors are symmetric, $V_0 = \sqrt{\gamma/2}(1 \ 1)$ and anti-symmetric $V_1 = \sqrt{\gamma/2}(1 \ -1)$, respectively (see main text). γ is the coupling rate of the channels to the cavity and therefore depends on the reflectivity at the interfaces of the waveguide. This yields

$$H_{\text{eff}} = \omega_0 \mathbb{1} + \frac{1}{2} \begin{pmatrix} -\delta\omega_0 - i\gamma & 0\\ 0 & \delta\omega_0 - i\gamma \end{pmatrix}$$
(S3)

The eigenvalues are then

$$\tilde{\omega}_{M\pm} = \omega_0 \pm \delta \omega_0 / 2 - i\gamma / 2 \tag{S4}$$

The scattering matrix $S(\omega)$ can then be derived from the effective Hamiltonian using $S(\omega) = 1 - iV^T[\omega \mathbb{1} - H_{\text{eff}}]^{-1}V$, where $V = [V_0V_1]$. The reflection coefficient $r(\omega)$ at the left interface is especially given by $r = 1 - iV_0^T[\omega \mathbb{1} - H_{\text{eff}}]^{-1}V_0$. Since

$$\left[\omega\mathbb{1} - H_{\text{eff}}\right]^{-1} = \begin{pmatrix} \frac{1}{\omega - \omega_0 + \delta\omega_0/2 + i\gamma/2} & 0\\ 0 & \frac{1}{\omega - \omega_0 - \delta\omega_0/2 + i\gamma/2} \end{pmatrix}$$
(S5)

we find

0

$$r(\omega) = 1 - \frac{i\gamma/2}{\omega - \omega_0 + \frac{1}{2}[\delta\omega_0 + i\gamma]} - \frac{i\gamma/2}{\omega - \omega_0 + \frac{1}{2}[-\delta\omega_0 + i\gamma]}$$
(S6)

This equation can then be simplified to

$$r(\omega) = \frac{[\omega - \omega_0]^2 + [\gamma^2 - \delta\omega_0^2]/4}{[\omega - \omega_0]^2 - [\delta\omega_0^2 + \gamma^2]/4 + i\gamma[(\omega - \omega_0]]}.$$
 (S7)

which is Eq. (3) of the main text.



FIG. S4. The transmission spectrum (blue line) and its reconstruction using the modal analysis (red dashed-line). The corresponding modes are shown in blue and orange. By compensating the absorption strength, $\Gamma_n \to \Gamma_n - \Gamma_a$ with $\Gamma_a = 30$ MHz, the maximum of the reconstructed transmission (black line) reaches unity. The dashed blue and orange lines are the contributions of the two modes when the absorption is removed.

Simple calculations show that the expression of $r(\omega)$ can also be factorized in terms of eigenfrequencies $\tilde{\omega}_{M\pm}$ and RL-eigenvalues $\tilde{\omega}_{R\pm} = \omega_0(z) \pm \frac{1}{2}\sqrt{\delta\omega_0(z)^2 - \gamma^2}$ as

$$r(\omega) = \frac{(\omega - \tilde{\omega}_{R+})(\omega - \tilde{\omega}_{R-})}{(\omega - \tilde{\omega}_{M+})(\omega - \tilde{\omega}_{M-})}.$$
 (S8)

As expected, $r(\omega)$ vanishes when $\omega = \tilde{\omega}_{R\pm}$.

Incorporating the impact of uniform absorption with the model is easily achieved by broadening the linewidth of resonances $\gamma \to \gamma + \Gamma_a$.

III. MODAL ANALYSIS

A. Harmonic Inversion

To extract the resonances $\tilde{\omega}_n = \omega_n - i\Gamma_n/2$ of the system, we perform a modal analysis between 7 and 11 GHz of the transmission coefficient. We seek to decompose $t(\omega)$ as

$$t(\omega) = \sum_{n} \frac{t_n}{\omega - \omega_n + i\Gamma_n/2}.$$
 (S9)

The coefficients t_n are the modal transmission coefficients that are associated with each resonance. We use the Harmonic Inversion (HI) method applied to the inverse Fourier transform of $t(\omega)$ in the time domain [1, 2]. Excellent agreement with experimental results is found with 19 modes.

Uniform losses within the sample broaden the linewidth, $\Gamma_n \rightarrow \Gamma_n + \Gamma_a$, and reduce transmission through the sample. We now estimate the absorption rate Γ_a at a RL-EP in the case of a perturbed Fabry-Perot interferometer with one dielectic slab placed at each interface. The maximum transmission found experimentally at the RL-EP is 0.865. We compensate the



FIG. S5. Spacing $\delta\omega_0(z)$ between central frequencies of the two resonances close to 8.2 GHz for a multimirror cavity with one dielectric slab on each interface. Near the EP at $z_{\rm EP} = 6.2$ mm, the experimental data (blue line) are in good agreement with a linear fit (red line) as a function of z.

linewidth in the reconstructed spectrum of $T(\omega)$) and find that the transmission reaches unity for $\Gamma_a = 30$ MHz, as shown in Fig.S4.

We observe that a slight asymmetry on the flattened line shape of the transmission spectrum found experimentally at the EP. The strengths of the two modes contributing to the transmission are indeed not equal. Their maxima are $T_n = |t_n|^2 = 1.536$ and $T_{n+1} = 1.398$. We attribute this difference to the joint effects of the imperfect symmetry of the system due to inevitable fabrication errors and a nonuniform absorption strength. Note that the modal strength exceeds unity as a consequence of the bi-orthogonality of quasi-normal modes in open systems [3]. In symmetric systems, T_n is equal to the Petermann factor of the modes, $T_n = K_n$, which is a measure of the degree of complexness of the eigenfunctions [4]. For isolated resonances, $K_n \sim 1$ but K_n increases with the modal overlap. Here, two resonances overlap yielding $T_n = K_n > 1.$

B. Spacing between resonant frequencies

In Fig. S5, the spacing between two resonant frequencies $\delta\omega_0(z)$ is shown as a function of the penetration depth z for a Fabry-Perot cavity with a a single alumina slab placed at each interface. Near the EP at $z_{\rm EP} = 6.8$ mm, $\delta\omega_0(z)$ is seen to be well approximated by a linear function. Using that $\delta\omega_0(z_{\rm EP}) =$ γ , we can write $\delta\omega_0(z) = \gamma + \kappa/2(z_{\rm EP} - z)$, where κ is a positive coefficient. The spacing between real RL-eigenvalues $\delta\tilde{\omega}_R = \sqrt{\delta\omega_0(z)^2 - \gamma^2}$ gives $\delta\tilde{\omega}_R =$ $\sqrt{\kappa\gamma(z_{\rm EP} - z) + \kappa^2/4(z_{\rm EP} - z)^2}$. In the vicinity of the EP, the square root behavior $\delta\tilde{\omega}_R \sim \sqrt{\kappa\gamma(z_{\rm EP} - z)}$ reflects the square-root sensitivity of EPs to an external perturbation of the system. The same is obtained for $z > z_{\rm EP}$ on the splitting of the imaginary parts of the two complex-conjugate RL-eigenvalues.



FIG. S6. Spectrum of reflection $R(\nu)$ at an EP for one dielectric slab at each interface in linear (a) and logarithmic (b) scales. The central frequency is $\nu_0 = 8.19$ GHz. The blue line corresponds to measured data, the red dashed line to the theory given by Eq. S7 with absorption, and the black line to the same equation but with the impact of absorption removed. The black dashed line displays a quadratic behavior.

C. Quartic behavior at EPs

In Fig. S6, we present the spectrum of the reflection at an EP in linear and log-scales. $R(\nu)$ is compared to Eq. 3 of the main text with and without including absorption. We include absorption using $\gamma \leftarrow \gamma + \Gamma_a$. In absence of absorption, the theory predicts a sharp dip at $\nu = \nu_0$. However, the minimum reflection increases with Γ_a . Measurements are in good agreement the theoretical prediction. Because the absorption strength is small, the quartic line shape which is a signature of EPs is preserved. The minimum reflection found experimentally is $R_{\rm min} \sim 10^{-3}$. We emphasize that even in the presence of absorption, the line shape is broadened relative to a quadratic behavior $R = (\omega - \omega_0)^2/\gamma^2$.

IV. MULTICHANNEL CAVITY

The reflection coefficient $r(\nu)$ and the transmission coefficient $t(\nu)$ are measured through a rectangular multichannel cavity (length L = 500 mm, width W = 250 mm and height H = 8 mm) between two arrays of N = 4single channels. Note that 8 transitions are seen on each



FIG. S7. (a) Photography of the Multi-channel cavity. (b) Sketch of the cavity for which the top has been removed to see the symmetric disorder inside the cavity.



FIG. S8. Colorscale representation of the last reflection eigenvalue $\tau_4(\nu, z)$ on a dB-scale for the multichannel case. The minimal reflection is shown for the left (top) and right (bottom) reflection matrices.

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side of the cavity in Fig. S7. However, only 4 of them are used to measure the reflection matrix. The others are not connected to the VNA. This open-circuit condition is similar to a metallic boundary conditions for the cavity.

An array of 14×20 holes are drilled symmetrically with respect to the center on the top plate of the cavity with a spacing of 12 mm between each hole (see Fig.S7(a)). Ten aluminium rods are inserted symmetrically through these holes thus creating a symmetric disorder inside the cavity (see Fig. S7(b)). As with the single mode waveguide presented in the main text, a perturbation is then introduced in the cavity by inserting symmetrically two aluminium rods spaced by 12 mm near to the center. The penetration depth of these two rods is tuned to observe the RL-EPs.

Due to its large dimensions and inevitable fabrication errors, the cavity is not perfectly symmetric. However, we verify that the RL-EP presented in the main text is found on both the left and the right reflection matrices. In Fig. S8, we present the colorscale representation of the smallest reflection eigenvalues $\tau_4(\nu, z)$ for both matrices. The multichannel RL-EP can be clearly observed on the two figures as expected.

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