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Lateral-Torsional Buckling of beams with warping restraints at Supports

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Abstract

In practice, the elastic lateral-torsional buckling of doubly symmetric I-section steel members is analyzed assuming free warping at supports. However, welded stiffeners, bolted end-plate connections or column base plates maintain elastically the opposite flange out-of-plane rotations and thus induce substantial warping restraints.

This paper proposes analytical formulations for the evaluation of the elastic critical bending moment for lateral-torsional buckling of beams taking into account warping restraints at supports. Assuming series of displacement and rotation fields containing one or two terms, the energy method permits to derive expressions of the critical bending moment under constant and linear distribution of the bending moment. A single expression of the warping coefficient k_w is proposed whatever the shape of the bending moment diagram. The factor C_1 can be conveniently calculated by multiplying two coefficients that depend on the shape of the bending moment diagram and on the warping restraint stiffness, respectively.

Finite Elements analyses of beams with warping restraints at supports have been performed considering a beam element model developed with LTBeamN and a shell element model created in ANSYS. The analytical model is in good agreement with the numerical results obtained by the two finite element models.

Keywords Elastic critical bending moment, Warping restraints, Lateral-torsional buckling.

1 Introduction

The elastic critical bending moment is commonly determined considering theoretical fork support conditions at both ends, i.e. fixed lateral displacement and torsional rotation but free warping. Because very few authors have proposed expressions for the critical moment of a beam elastically restraint against warping, the influence of the warping restraints on the critical moment is usually neglected.

In this paper, the influence of warping restraints on the critical bending moment is investigated. After a brief review of the literature about the determination of the elastic critical moment of a uniform beam with doubly symmetric cross-section and with warping restraints at supports (see Figure 1), an analytical model is described. Considering appropriate series of displacement and rotation fields, the energy method is used to derive a formula of the elastic critical bending moment under constant and linear distribution of the bending moment. It is shown that the use of displacement and rotation fields approximated with one term is correct for uniform bending moment but for a linear moment diagram two terms are necessary.

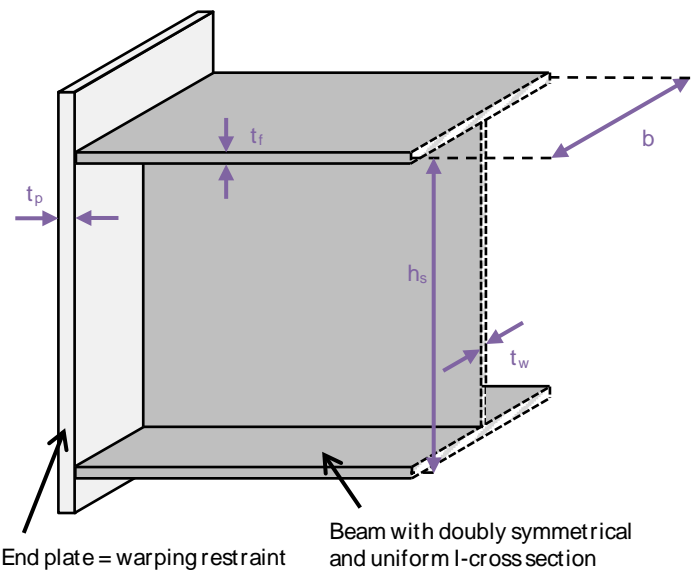


Figure 1 End of a beam with plates restraining warping

The derived expression for the critical bending moment depends on

the warping stiffness through the coefficients C_1 and k_w . While the warping coefficient k_w has a single expression, C_1 also depends on the shape of the bending moment distribution.

Finally, the predictions of the analytical model are compared against finite element results of beams with warping restraints at supports, computed either using beam elements model with the software LTBeamN [1] or shell elements model with ANSYS. A good agreement between the analytical model and finite element analyses is observed.

2 Literature overview

In the French National Annex to Eurocode 3 Part 1-1 [2], the formula of the elastic critical bending moment can account for warping restraints only through the warping coefficient k_w . No expression is given to determine k_w considering potential warping restraints. The critical moment of a beam with doubly symmetric cross-section and with free out-of-plane rotation considering a linear distribution of the bending moment is:

$$M_{cr} = C_1 \frac{\pi^2 EI_z}{L^2} \left\{ \sqrt{\left(\frac{1}{k_w}\right)^2 \frac{I_w}{I_z} + \frac{GI_T L^2}{EI_z \pi^2}} \right\} \quad (1)$$

Where:

- C_1 : Equivalent uniform moment factor
- E : Modulus of elasticity
- I_z : z-axis second moment of area (see Figure 2)
- L : distance between lateral restraints
- I_w : Warping constant
- I_T : Torsional constant
- G : Shear modulus.

The warping coefficient k_w varies between 0,5 when warping is fixed to 1 when warping is free. However, no expression is given to compute k_w and it is assumed that C_1 depends only on the bending moment diagram without any influence of the warping restraints.

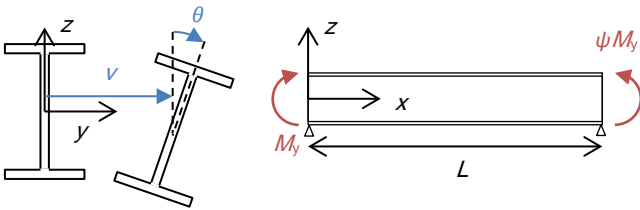


Figure 2 Axes and notations for the analytical derivation

Based on the work of Lindner and Gietzelt [3], it has been suggested in [4] to evaluate k_w for I-beams with end plates at both ends using the following expression:

$$k_w = 1 - \frac{0.5}{1 + \frac{2EI_w}{c_w L}} \quad (2)$$

$$c_w = \frac{1}{3} G b t_p^3 h_s \quad (3)$$

Where:

- c_w : Stiffness of the warping restraints (see Figure 3)
- t_p : Thickness of the end plates
- h_s : Distance between flanges centres.

Therefore, the warping coefficient depends on the warping stiffness

EI_w , the length L of the beam, and on the stiffness of the warping restraint c_w . The bending moment distribution assumed for expression (2) is not specified. No expression is given for C_1 ; the values given in the French National Annex [2] considering free warping at supports should therefore be used.

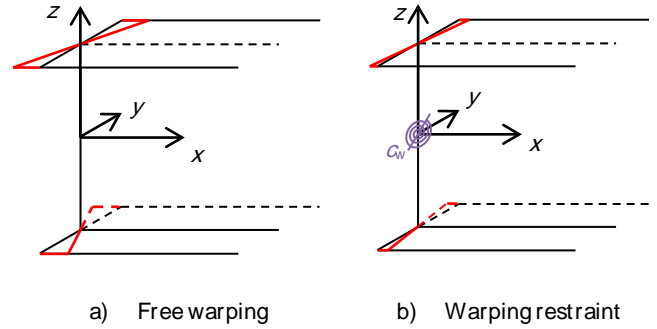


Figure 3 Flanges longitudinal displacements when warping is a) free and b) restrained

For a uniform bending moment distribution, Pi and Trahair [5] proposed the following approximation for k_w :

$$k_w = \frac{6,5 + \frac{c_w L}{EI_w}}{6,5 + 2,25 \frac{c_w L}{EI_w}} \quad (4)$$

Details on the derivation of expressions (2) to (4) in references [3]-[5] are missing. Piotrowski and Szychowski [6] used the energy method to propose a general expression of the elastic critical moment accounting for warping restraints. The out-of-plane displacement v and torsional rotation θ are approximated using power polynomials. The proposed expression for the critical moment of a beam under linear bending moment with free out of plane rotation is:

$$M_{cr} = D_1 \frac{\sqrt{EI_z (C_1 GI_T L^2 + C_2 EI_w)}}{C_3 L^2} \quad (5)$$

The values of C_1 , C_2 , C_3 and D_1 , depending on the ratio ψ between end moments, are given in Table 1.

Table 1 Coefficients to determine the critical moment

$-0,5 < \psi \leq -1$	
C_1	$57,6(1,457 - 2,4\kappa_w + \kappa_w^2)$
C_2	$691,2(1,2 - \kappa_w)$
C_3	$\sqrt{(1 + \psi)^2(2,124 - 3,497\kappa_w + 1,440\kappa_w^2)}$
D_1	$1,018 + 0,000297\kappa_w^2 - 0,009\kappa_w^6 - e^{-2,5\psi}(0,108 - 0,001\kappa_w - 0,014\kappa_w^6)$
$-1 \leq \psi \leq -0,5$	
C_1	$345,75(1,457 - 2,4\kappa_w + \kappa_w^2)$
C_2	$4148,98(1,2 - \kappa_w)$
C_3	$\sqrt{(1 - \psi)^2(1,681 - 2,880\kappa_w + 1,235\kappa_w^2)}$
D_1	$1,068 + 0,014\kappa_w^2 - e^{3\psi}(1,704 + 0,099\kappa_w + 0,215\kappa_w^6)$

The index of fixity against warping κ_w varies between 0 when warping is free, and 1 when warping is fixed. It can be obtained by the following expression:

$$\kappa_w = \frac{1}{1 + \frac{2EI_w}{c_w L}} \quad (6)$$

From expression (5), the warping coefficient is identified as :

$$k_w = \frac{\pi}{2\sqrt{3}} \sqrt{\frac{1,457 - 2,4\kappa_w + \kappa_w^2}{1,2 - \kappa_w}} \quad (7)$$

The coefficient C_1 (when referring to equation (1)) is extracted from expression (5) and depends on the bending moment distribution, through the ratio ψ , and the warping restraints, through κ_w . Some values of C_1 are given in Table 2 according to expression (5) and according to TableM.1 from the French National Annex to Eurocode 3 Part 1-1 [2].

Table 2 highlights the evolution of the coefficient C_1 with bending moment distribution and warping restraints. The values of C_1 when warping is fully restrained increase from 10% when the bending moment is constant to more than 30% when ψ is equal to -1 when compared to free warping.

Besides, the values of C_1 given by the French National Annex [2] are lower than those given by Piotrowski [6] for free warping. Finite element analyses should be performed to assess the safety level of the approaches of the French National Annex and of Piotrowski.

Table 2 C_1 for different values of ψ

Reference	κ_w	ψ				
		1	0,5	0	-0,5	-1
French NA [2]	0	1,00	1,31	1,77	2,33	2,55
	∞	1,010	1,317	1,822	2,527	2,709
Piotrowski [6]	0	1,114	1,458	2,040	3,010	3,566
	∞	1,114	1,458	2,040	3,010	3,566

Simple expressions have been proposed to take into account the effect of warping restraints on the elastic critical moment, mainly through the warping coefficient k_w . However, expressions (2) and (4) seem to be valid only for beams under uniform bending moment. Piotrowski [6] proposed to account for the bending moment distribution but the expressions are complex for a daily use in design office. Expressions based on consistent derivation of an analytical model are yet to be developed.

3 Analytical Model

3.1 Introduction

The energy method is used to derive an analytical expression of the elastic critical bending moment of a beam with warping restraints at supports. Firstly, the model is defined in the simple case of a beam under uniform bending moment in §3.2, using one term for the out-of-plane displacement v and the torsional rotation θ fields. A second term is then added to enhance the displacement and rotation fields in presence of a linear distribution of the bending moment in §3.3.

The analytical model relies on the following assumptions:

- Uniform beam with doubly symmetrical I-cross-section
- Fork supports conditions
- Warping restraints at supports (stiffness c_w).

Figure 2 present the coordinate system, consistent with Eurocode 3 Part 1-1 [7] and notations used to develop the analytical model.

3.2 Uniform bending moment distribution

3.2.1 Displacements and rotations fields

For a beam under constant bending moment $M_{y,0}$ with free warping and out-of-plane rotation at both supports, the exact solution of the lateral displacement and the torsional rotation has the shape of half of a sine-wave:

$$v(x) = v_0 \sin\left(\frac{\pi x}{L}\right) \quad (8)$$

$$\theta(x) = \theta_0 \sin\left(\frac{\pi x}{L}\right) \quad (9)$$

Where:

- v_0 : Magnitude of the lateral displacement
- θ_0 : Magnitude of the torsional rotation.

However, for fixed warping at supports, obtaining the exact expression of the torsional rotation is not obvious. Vlasov [8] and Djalaly [9] have proposed the following approximation, considering a cosine-wave:

$$\theta(x) = \theta_0 \left[1 - \cos\left(2\frac{\pi x}{L}\right)\right] \quad (10)$$

Elastic warping restraints at beam ends induce the following boundary conditions between the bimoment B the warping stiffness c_w and the torsional rotation:

$$B(0) = -c_w \frac{d\theta}{dx}(0) \quad \text{and} \quad B(L) = c_w \frac{d\theta}{dx}(L) \quad (11)$$

Besides, the bimoment is connected to the beam's warping stiffness and torsional rotation by:

$$B(x) = -EI_w \frac{d^2\theta}{dx^2}(x) \quad (12)$$

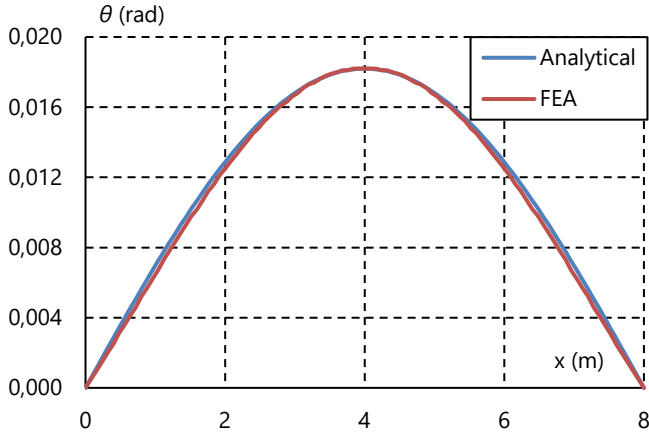
The rotation has to comply with the following kinematic boundary conditions at supports, obtained by combining expressions (11) and (12):

$$\frac{d^2\theta}{dx^2}(0) = \frac{c_w}{EI_w} \frac{d\theta}{dx}(0) \quad \text{and} \quad \frac{d^2\theta}{dx^2}(L) = -\frac{c_w}{EI_w} \frac{d\theta}{dx}(L) \quad (13)$$

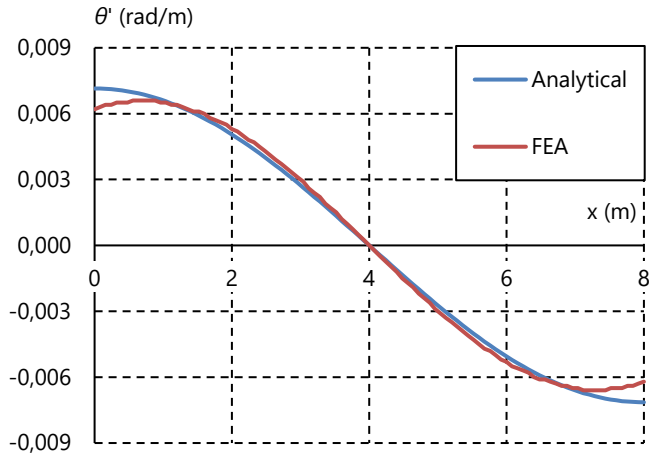
An approximate distribution of the torsional rotation is developed from equations (9) and (10), in agreement with boundary conditions (13):

$$\theta(x) = \theta_0 \left\{ 4\pi \sin\left(\frac{\pi x}{L}\right) + \frac{c_w L}{EI_w} \left[1 - \cos\left(2\frac{\pi x}{L}\right)\right] \right\} \quad (14)$$

The torsional rotation distribution and its first derivative are compared to finite element analyses. Figure 4 shows the results given by equation (14) and the results of an analysis performed with LTBeamN [1] using beam elements. The example correspond to a 8 m-span beam under uniform bending moment with a 800x8 web, 200x14 flanges and end-plates of 30 mm thickness.



a) $\theta(x)$



b) $\theta'(x)$

Figure 4 a) Rotation and b) first derivative along a beam with warping restraints under constant bending moment

The analytical and finite element models give very close results. Some discrepancies appear when looking at the derivative at supports, explaining that small differences might be encountered between the analytical model to be developed and the numerical model.

3.2.2 Potential energy

The energy method is based on the minimisation of the potential energy Π , defined as the difference between the elastic strain energy U and the work of external loads W that can be expressed as (see Timoshenko [10]):

$$\delta^2 \Pi = \delta^2 (U - W) = 0 \quad (15)$$

The total potential energy is a function of the magnitudes v_0 and θ_0 . Minimisation of the total potential energy is obtained by equating to 0 its partial derivatives. Expression (15) then rewrites as:

$$\begin{Bmatrix} \frac{\partial \Pi}{\partial v_0} \\ \frac{\partial \Pi}{\partial \theta_0} \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix} \begin{Bmatrix} v_0 \\ \theta_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (16)$$

Considering the previous assumptions described in 3.1, the total potential energy is given by:

$$\Pi = U_{beam} + U_{spring} - W \quad (17)$$

With:

$$U_{beam} = \frac{1}{2} \int (EI_z v_{,xx}^2 + EI_w \theta_{,xx}^2 + GI_t \theta_{,x}^2) dx$$

$$U_{spring} = c_w \frac{\theta'(0)^2 + \theta'(L)^2}{2}$$

$$W = - \int (M_{y,0} \theta v_{,xx}) dx$$

Where “,x” stands for the first derivative by x and “,xx” for the second.

The contribution of the in-plane displacement is generally negligible when compared to that of the out-of-plane displacement and torsional rotation. This contribution is therefore omitted in the expression (17) of the potential energy. Replacing the lateral displacement and the torsional rotation by expressions (8) and (14), the potential energy becomes:

$$\begin{aligned} \Pi = \frac{1}{2} \left(\frac{\pi}{L} \right)^2 & \left[EI_z \left(\frac{\pi}{L} \right)^2 v_0^2 \frac{L}{2} + EI_w \theta_0^2 \left(\frac{\pi}{L} \right)^2 \xi_w + GI_t \theta_0^2 \xi_t \right. \\ & \left. - 2M_{y,0} \theta_0 v_0 \sqrt{2L} \xi_M \right] + \frac{\theta_0^2}{2} \left(\frac{\pi}{L} \right)^2 \xi_s \end{aligned} \quad (18)$$

With:

$$\xi_M = \pi \sqrt{2L} + \frac{4\sqrt{2}\sqrt{L} c_w L}{3 \pi EI_w}$$

--

$$\xi_t = 8\pi^2 L + 2L \left(\frac{c_w L}{EI_w} \right)^2 + \frac{64}{3} L \frac{c_w L}{EI_w}$$

--

$$\xi_w = \xi_t + 6L \left(\frac{c_w L}{EI_w} \right)^2$$

--

$$\xi_s = 32c_w \pi^2$$

3.2.3 Elastic critical moment

Taking the derivative of Π with respect to v_0 and θ_0 , we obtain two equations:

$$EI_z \left(\frac{\pi}{L} \right)^4 v_0 \frac{L}{2} - M_{y,0,cr} \theta_0 \left(\frac{\pi}{L} \right)^2 \xi_M \sqrt{2L} = 0 \quad (19)$$

$$EI_w \theta_0 \left(\frac{\pi}{L} \right)^2 \xi_w + GI_t \theta_0 \xi_t - M_{y,0,cr} v_0 \xi_M \sqrt{2L} + \theta_0 \xi_s = 0 \quad (20)$$

From equation (19) arises the relationship between the two magnitudes:

$$v_0 = \theta_0 \frac{M_{y,0,cr}}{EI_z \left(\frac{\pi}{L} \right)^2} 2 \sqrt{\frac{2}{L}} \xi_M \quad (21)$$

The elastic critical bending moment $M_{y,0,cr}$ is then derived from expressions (20) and (21):

$$M_{y,0,cr} = \sqrt{\frac{\xi_t}{\xi_M^2} EI_z \left(\frac{\pi}{L} \right)^2} \sqrt{\frac{I_w \xi_w}{I_z \xi_t} + \frac{\xi_s L^2}{\xi_t EI_z \pi^2} + \frac{GI_t L^2}{EI_z \pi^2}} \quad (22)$$

From the previous expression, the warping coefficient k_w and the equivalent uniform moment factor C_1 are identified as:

$$k_w = \sqrt{\frac{\pi^2 + \frac{8}{3} \frac{c_w L}{EI_w} + \left(\frac{1}{2} \frac{c_w L}{EI_w} \right)^2}{\pi^2 + \frac{20}{3} \frac{c_w L}{EI_w} + \left(\frac{c_w L}{EI_w} \right)^2}} \quad (23)$$

$$C_1 = \frac{\sqrt{\pi^2 + \frac{8c_w L}{3EI_w} + \left(\frac{1c_w L}{2EI_w}\right)^2}}{\pi + \frac{4c_w L}{3\pi EI_w}} \quad (24)$$

The elastic critical bending moment of a doubly symmetric beam under constant moment with warping restraints at supports is given by expression (1) with k_w and C_1 given by (23) and (24).

3.3 Linear bending moment distribution

Considering a beam subjected to a linear bending moment distribution with ψ being the ratio between end moments, we derive the expression of the critical bending moment based on the out-of-plane displacement and torsional rotation given by (8) and (14). The following expressions are obtained for coefficients k_w and C_1 :

$$C_1 = \frac{2}{1 + \psi} \frac{\sqrt{\pi^2 + \frac{8c_w L}{3EI_w} + \left(\frac{1c_w L}{2EI_w}\right)^2}}{\pi + \frac{4c_w L}{3\pi EI_w}} \quad (25)$$

$$k_w = \frac{\sqrt{\pi^2 + \frac{8c_w L}{3EI_w} + \left(\frac{1c_w L}{2EI_w}\right)^2}}{\sqrt{\pi^2 + \frac{20c_w L}{3EI_w} + \left(\frac{c_w L}{EI_w}\right)^2}} \quad (26)$$

Expression (26) for k_w is the same as (23) derived under constant bending moment. Therefore, the warping coefficient does not depend on the bending moment distribution. Besides, C_1 can be expressed as the product of two distinct factors depending on ψ and on the warping stiffness, respectively. The latter is identical to expression (24) of C_1 obtained for a uniform bending moment.

For free warping at supports, it is clear that expression (25) leads to inconsistencies for negative values of ψ . Indeed, when the ratio tends towards -1, C_1 tends towards the infinity. This discrepancy comes from the displacement and rotation fields used to derive expressions (25) and (26). Equations (8) and (14) are well suited when the bending moment is constant, or if $\psi > 0.5$. Therefore, the displacement and rotation fields need to be enriched with a second term for v and θ .

The general expression of the out-of-plane displacement of a beam with free out-of-plane rotation at both supports can be expressed by the following sum:

$$v(x) = \sum_{i=1}^{\infty} v_i \sin\left(i \frac{\pi x}{L}\right) \quad (27)$$

To avoid a too cumbersome analytical development, we will consider only the first two terms of the series. The out-of-plane displacement is finally given by (29).

A second term is also added to the torsional rotation, fulfilling the kinematical boundary conditions given by (13). The number of sine waves along the beam is increased, giving the following second term:

$$\theta_2(x) = \theta_2 \left\{ \frac{16}{3} \pi \sin\left(3 \frac{\pi x}{L}\right) + \frac{c_w L}{EI_w} \left[1 - \cos\left(\frac{4\pi x}{L}\right) \right] \right\} \quad (28)$$

This second term varies between $\sin\left(3 \frac{\pi x}{L}\right)$ when warping is free and $\left[1 - \cos\left(\frac{4\pi x}{L}\right) \right]$ when warping is prevented at both ends. Finally, the new displacement and rotation fields are:

$$v(x) = v_1 \sin\left(\frac{\pi x}{L}\right) + v_2 \sin\left(2 \frac{\pi x}{L}\right) \quad (29)$$

$$\theta(x) = \theta_1 \left\{ 4\pi \sin\left(\frac{\pi x}{L}\right) + \frac{c_w L}{EI_w} \left[1 - \cos\left(\frac{2\pi x}{L}\right) \right] \right\} + \theta_2 \left\{ \frac{16}{3} \pi \sin\left(3 \frac{\pi x}{L}\right) + \frac{c_w L}{EI_w} \left[1 - \cos\left(\frac{4\pi x}{L}\right) \right] \right\} \quad (30)$$

From expressions (29) and (30), the potential energy Π of a doubly symmetric beam subjected to end moments with warping restraints at supports is calculated. The critical moment M_{cr} is then obtained by solving the following equation:

$$\begin{bmatrix} \frac{\partial \Pi}{\partial v_1} \\ \frac{\partial \Pi}{\partial v_2} \\ \frac{\partial \Pi}{\partial \theta_1} \\ \frac{\partial \Pi}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} K_{11} & 0 & M_{cr} K_{13} & M_{cr} K_{14} \\ 0 & K_{22} & M_{cr} K_{23} & M_{cr} K_{24} \\ M_{cr} K_{13} & M_{cr} K_{23} & K_{33} & K_{34} \\ M_{cr} K_{14} & M_{cr} K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (31)$$

With:

$$\begin{aligned} K_{11} &= EI_z \left(\frac{\pi}{L}\right)^4 \frac{L}{2} \\ -- \\ K_{22} &= 8EI_z \left(\frac{\pi}{L}\right)^4 L \\ -- \\ K_{33} &= GI_t \left(\frac{\pi}{L}\right)^2 \xi_{t,1} + EI_w \left(\frac{\pi}{L}\right)^4 \xi_w + \left(\frac{\pi}{L}\right)^2 \xi_s \\ -- \\ K_{44} &= GI_t \left(\frac{\pi}{L}\right)^2 \xi_{t,2} + EI_w \left(\frac{\pi}{L}\right)^4 \xi_{w,2} + 16 \left(\frac{\pi}{L}\right)^2 \xi_s \\ -- \\ K_{13} &= - \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{L}{2}} \xi_{M,1} (1 + \Psi) \\ -- \\ K_{14} &= - \left(\frac{\pi}{L}\right)^2 \frac{16L}{15\pi} \frac{c_w L}{EI_w} (1 + \Psi) \\ -- \\ K_{23} &= \left(\frac{\pi}{L}\right)^2 \frac{L}{\pi} \xi_{M,2} (\Psi - 1) \\ -- \\ K_{24} &= \left(\frac{\pi}{L}\right)^2 \frac{L}{\pi} \xi_{M,3} (\Psi - 1) \\ -- \\ K_{34} &= -GI_t \frac{\pi^2 c_w L}{L EI_w} \frac{256}{15} - EI_w \left(\frac{\pi}{L}\right)^2 \frac{\pi^2 c_w L}{L EI_w} \frac{3328}{15} + 4 \left(\frac{\pi}{L}\right)^2 \xi_s \\ -- \\ \xi_{t,2} &= 128\pi^2 L + \frac{1024}{7} L \frac{c_w L}{EI_w} + 8L \left(\frac{c_w L}{EI_w}\right)^2 \\ -- \\ \xi_{w,2} &= 9\xi_{t,2} + 56L \left(\frac{c_w L}{EI_w}\right)^2 \\ -- \\ \xi_{M,2} &= \frac{128}{9} + \frac{3c_w L}{2EI_w} \\ -- \\ \xi_{M,3} &= \frac{512}{25} + \frac{8c_w L}{3EI_w} \end{aligned}$$

The critical bending moment is finally derived by equalling to zero the determinant of the 4x4 matrix in equation (31). The coefficients k_w and C_1 are then identified in the 4th degree equation on M_{cr} .

The expression of the warping coefficient k_w is similar to that derived with a constant bending moment (23). The equation on M_{cr} is

then simplified as an equation on C_1 . To simplify, we assume that the ratio $\sqrt{EI_w}/GI_tL^2$ is null. It leads to the minimum values of C_1 and therefore provides conservative values. The equation becomes:

$$AC_1^4 - BC_1^2 + C = 0 \quad (32)$$

With:

$$A = (\Psi^2 - 1)^2 \left[40856 + 18612 \frac{c_w L}{EI_w} + 3406 \left(\frac{c_w L}{EI_w} \right)^2 + 290,1 \left(\frac{c_w L}{EI_w} \right)^3 + 9,51 \left(\frac{c_w L}{EI_w} \right)^4 \right]$$

--

$$B = \pi^2 \left\{ 8(\Psi + 1)^2 \left[\frac{\pi^2}{2L^2} \xi_{M,2}^2 \xi_{t,2} + \left(\frac{16 c_w L}{15 EI_w} \right)^2 \left(\frac{\xi_t}{L} + 16\pi \sqrt{\frac{2}{L}} \xi_M \right) \right] + \frac{(\Psi-1)^2}{2} \left[\frac{\xi_{M,2}^2 \xi_{t,2} + \xi_{M,3}^2 \xi_t}{L} + \frac{512 c_w L}{15 EI_w} \xi_{M,2} \xi_{M,3} \right] \right\}$$

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$$C = 4\pi^4 \left\{ \frac{\xi_t \xi_{t,2}}{L^2} - \left(\frac{256 c_w L}{15 EI_w} \right)^2 \right\}$$

The positive solution of the previous equation is:

$$-1 < \Psi < 1 \quad C_1 = \sqrt{\frac{4\pi^2 \xi_1}{(\Psi^2 - 1)^2} \left[\Delta - \sqrt{\Delta^2 - \frac{(\Psi^2 - 1)^2}{4\pi^2 \xi_1 \xi_3}} \right]} \quad (33)$$

$$\Delta = (\Psi + 1)^2 + \xi_2 (\Psi - 1)^2$$

$$\Psi = 1 \quad C_1 = \frac{1}{\sqrt{8\xi_3}} \quad (34)$$

$$\Psi = -1 \quad C_1 = \frac{1}{\sqrt{8\xi_2 \xi_3}} \quad (35)$$

With:

$$\xi_1 = \frac{123058 + 47499 \frac{c_w L}{EI_w} + 7324 \left(\frac{c_w L}{EI_w} \right)^2 + 543,4 \left(\frac{c_w L}{EI_w} \right)^3 + 16,50 \left(\frac{c_w L}{EI_w} \right)^4}{40856 + 18612 \frac{c_w L}{EI_w} + 3406 \left(\frac{c_w L}{EI_w} \right)^2 + 290,1 \left(\frac{c_w L}{EI_w} \right)^3 + 9,51 \left(\frac{c_w L}{EI_w} \right)^4}$$

--

$$\xi_2 = \frac{18041 + 6938 \frac{c_w L}{EI_w} + 1049 \left(\frac{c_w L}{EI_w} \right)^2 + 73,57 \left(\frac{c_w L}{EI_w} \right)^3 + 2,01 \left(\frac{c_w L}{EI_w} \right)^4}{123058 + 47499 \frac{c_w L}{EI_w} + 7324 \left(\frac{c_w L}{EI_w} \right)^2 + 543,4 \left(\frac{c_w L}{EI_w} \right)^3 + 16,50 \left(\frac{c_w L}{EI_w} \right)^4}$$

--

$$\xi_3 = \frac{123058 + 47499 \frac{c_w L}{EI_w} + 7324 \left(\frac{c_w L}{EI_w} \right)^2 + 543,4 \left(\frac{c_w L}{EI_w} \right)^3 + 16,50 \left(\frac{c_w L}{EI_w} \right)^4}{984463 + 379988 \frac{c_w L}{EI_w} + 59097 \left(\frac{c_w L}{EI_w} \right)^2 + 4572 \left(\frac{c_w L}{EI_w} \right)^3 + 157,9 \left(\frac{c_w L}{EI_w} \right)^4}$$

C_1 depends on the ratio ψ and on the warping stiffness through the parameters ξ_i . In the specific cases of free and fixed warping at both supports, the values of ξ_1 , ξ_2 and ξ_3 are given in Table 3.

Table 3 Values of ξ_1 , ξ_2 and ξ_3 when free or fixed warping

	ξ_1	ξ_2	ξ_3
Free warping ($c_w = 0$)	3,012	0,147	0,125
Fixed warping ($c_w \rightarrow \infty$)	1,735	0,122	0,104

The values of C_1 are calculated in Table 4 for various values of ψ in

the two extreme cases of free and fixed warping at supports. We can notice an increase in the values of C_1 from 9,4 to almost 20 % when warping is fixed.

Values of C_1 given by Table M.1 from the French National Annex to Eurocode 3 Part 1-1 [2] are also presented in Table 4 for free warping. The analytical model give results close to the French NA even though small differences, below 15%, can be noticed. Again, like in Table 2 where Piotrowski's [6] results were found to be greater than the French NA when warping is free, finite element analyses must be computed in order to assess the safety level of the two approaches.

Table 4 C_1 for free or fixed warping at supports

ψ	C_1		$C_{1, \text{fixed warp.}} / C_{1, \text{free warp.}}$	C_1 French NA [2]
	Free warping	Fixed warping		
1	1,00	1,09	1,094	1,00
0,75	1,14	1,25	1,094	1,14
0,5	1,32	1,45	1,096	1,31
0,25	1,56	1,72	1,102	1,52
0	1,88	2,10	1,114	1,77
-0,25	2,28	2,60	1,141	2,05
-0,5	2,67	3,14	1,179	2,33
-0,75	2,81	3,37	1,198	2,57
-1	2,61	3,13	1,198	2,55

In Figure 5, the warping coefficient k_w is plotted against the parameter $\frac{c_w L}{EI_w}$ accounting for the warping restraints. It shows the results obtained with the proposals of Lindner and Gietzelt [3] and ECCS [4], Pi and Trahair [5], Piotrowski and Szychowski [6] and expression (23).

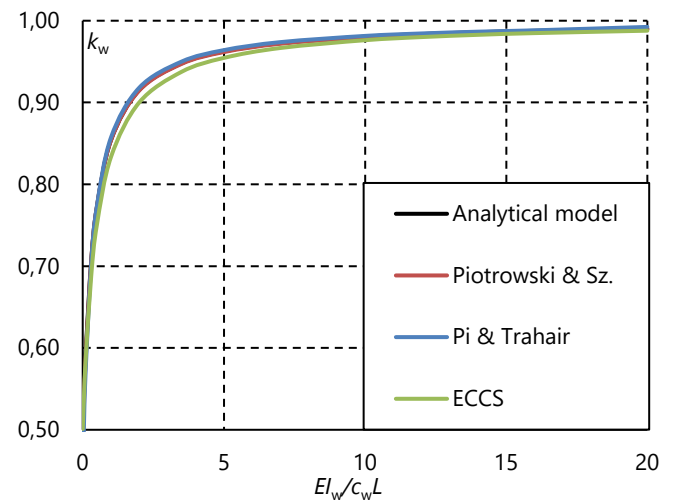


Figure 5 Warping coefficient k_w against $\frac{EI_w}{c_w L}$

The different proposals give close results. Some differences appear when the restraints tend to fix warping, i.e. when $c_w \rightarrow \infty$. Indeed, Pi

and Trahair tend towards 0,44 while the other proposals tend towards 0,5.

From the analytical model developed in §3, expressions of the equivalent uniform moment factor C_1 and the warping coefficient k_w were derived to determine the elastic critical bending moment of a uniform beam with doubly symmetric I-section subjected to linear bending moment distribution with warping restraints at supports. Whatever the bending moment distribution, k_w only depends on the warping stiffness, as expressed in equation (23). However, C_1 depends on both the warping stiffness and the bending moment distribution as expressions (33) to (35) state. Formulae derived from the analytical model will be compared against finite element analyses in §4.

4 Finite Element Analyses

4.1 Numerical Models

Two finite element models are used for comparisons:

- Shell elements with ANSYS
- Beam elements with LTBeamN [1].

Linear Bifurcation Analyses (LBA) are performed considering Young's modulus $E=210\,000$ MPa and Poisson's ratio $\nu=0,3$.

The geometry of the studied beams is given in Table 5. Warping is restrained by end plates, fixed or free at both ends. Thickness of the end plates varies between 7 and 70 mm to extend the study beyond the common practice.

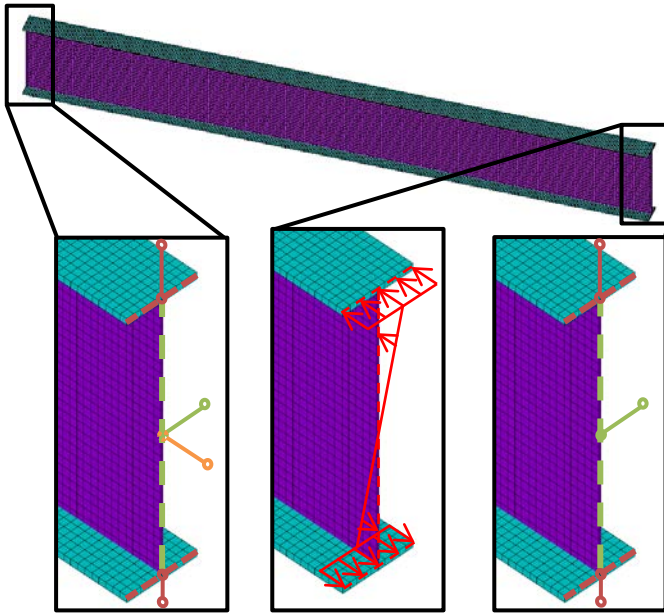


Figure 6 End moment and fork support conditions

The shell element model is developed using 8-node elements, with 6 degrees of freedom at each node. The out-of-plane displacement and torsional rotation of the section are fixed at supports as well as the axial displacement of web centre at one end to reproduce typical fork support conditions (see Figure 6). End plates are modelled at supports with shell elements. In addition, distortion is prevented along the beam.

Eventually, nodal forces along the x-axis are applied to provoke end moments, leading to a linear bending moment distribution as shown in Figure 6.

Figure 7 exhibits the out-of-plane displacements corresponding to the critical mode of beam P2 subjected to a) uniform and b) linear bending moment distribution with $\psi = -1$, along with 20 mm-thick end plates. The two modal shapes are quite different. Under constant bending moment, half of a sine wave is visible while a complete sine-wave appear for $\psi = -1$. These observations are consistent with the displacement fields assumed in §3.3.

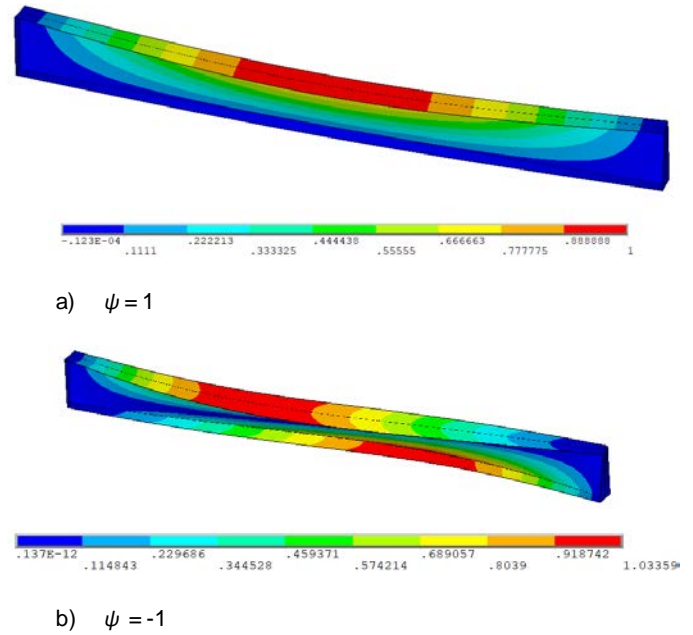


Figure 7 Out-of-plane displacement of P2 with 20 mm-thick end plates

The beam element model is developed with two-noded beam elements with 7 degrees of freedom (including warping) at each node. Fork support conditions are enforced by preventing lateral displacement and torsional rotation at both ends and longitudinal displacement at one end.

The degree of freedom corresponding to warping may be fixed or connected to a warping spring with a stiffness c_w , calculated according to equation (3). End moments are directly applied to induce a linear bending moment distribution.

Table 5 Geometry of studied beams

Beam	h_s (mm)	t_w (mm)	b (mm)	t_r (mm)	L (m)
P1	600	6	200	14	6
P2	800	10	280	14	10
P3	800	6	280	12	10
P4	300	4	100	6	6 & 9
P5	300	6	100	10	6 & 9
P6	347,3	8	170	12,7	6
P7	337,5	12,5	300	22,5	3 & 6

4.2 Comparison with the Analytical Model

4.2.1 Equivalent uniform moment factor C_1

Finite element analyses have been performed with the beam model using LTBeamN and compared against analytical predictions. The

coefficient C_1 was determined for a beam under linear bending moment distribution in the cases of (i) free warping and (ii) fixed warping at both ends and compared against both analytical model and the French National Annex [2].

Analyses were performed on beams P1, P2, P3, P6 and P7 (see Table 5). Figure 8 presents the values of C_1 as a function of ψ .

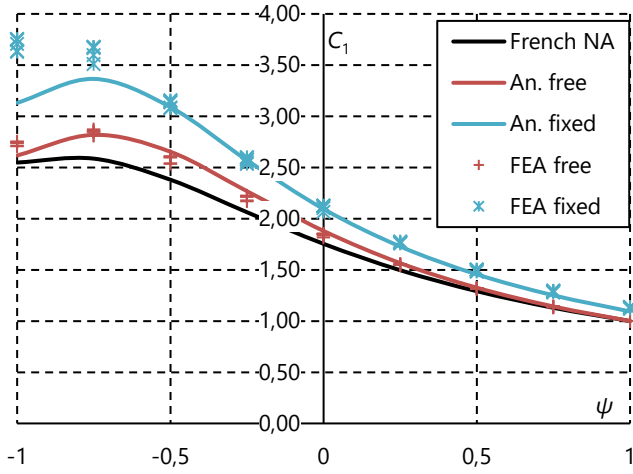


Figure 8 C_1 for free or fixed warping

In addition to the values given in the French National Annex [2], the figure shows the results obtained with finite element analyses (FEA) and the analytical model, i.e. expressions (33) to (35), in the cases of free warping (referred to as “free”) and fixed warping (referred to as “fixed”).

For free warping, the analytical model is in good agreement with FEA results. The value of C_1 might be slightly overestimated (of less than 3%) when ψ is between -0,5 and 0 but the French NA is quite conservative: C_1 may be underestimated up to 10%.

For fixed warping, Figure 8 shows a good agreement between the numerical and analytical models when ψ is greater than -0,75. When the ratio is below that value, the analytical model lays on the safe side, 15% lower than FEA results. The deviation between the values of C_1 given by the French NA and FEA results when warping is fixed is up to 45%, confirming that the French NA is overly conservative when warping is fixed.

4.2.2 Elastic critical bending moment

To state about the consistency of the complete analytical model, analyses were conducted using the two finite element models. The values of the elastic critical bending moment M_{cr} resulting from FE analyses were then compared against predictions of the analytical model as well as Piotrowski’s [6] proposals.

The ratio $M_{cr}/M_{cr,0}$, with $M_{cr,0}$ being the critical bending moment when warping is free, is plotted against the end plates’ thickness for beam P2 in Figure 9a) to c). Thickness t_p of the end plates is directly connected to the warping stiffness c_w given by expression (3).

A jump in the values of t_p can be seen on Figure 9a) to c), between a 70mm thickness and infinite thickness corresponding to fixed warping conditions. The analytical model depicted in this article and formula (5) from Piotrowski [6] are represented by continuous curves for thicknesses ranging between 0 and 70mm and by empty triangles for fixed warping.

Figure 9 shows a significant increase of the critical bending moment

when warping is fixed. In every cases, its value is at least almost doubled (+95% or more).

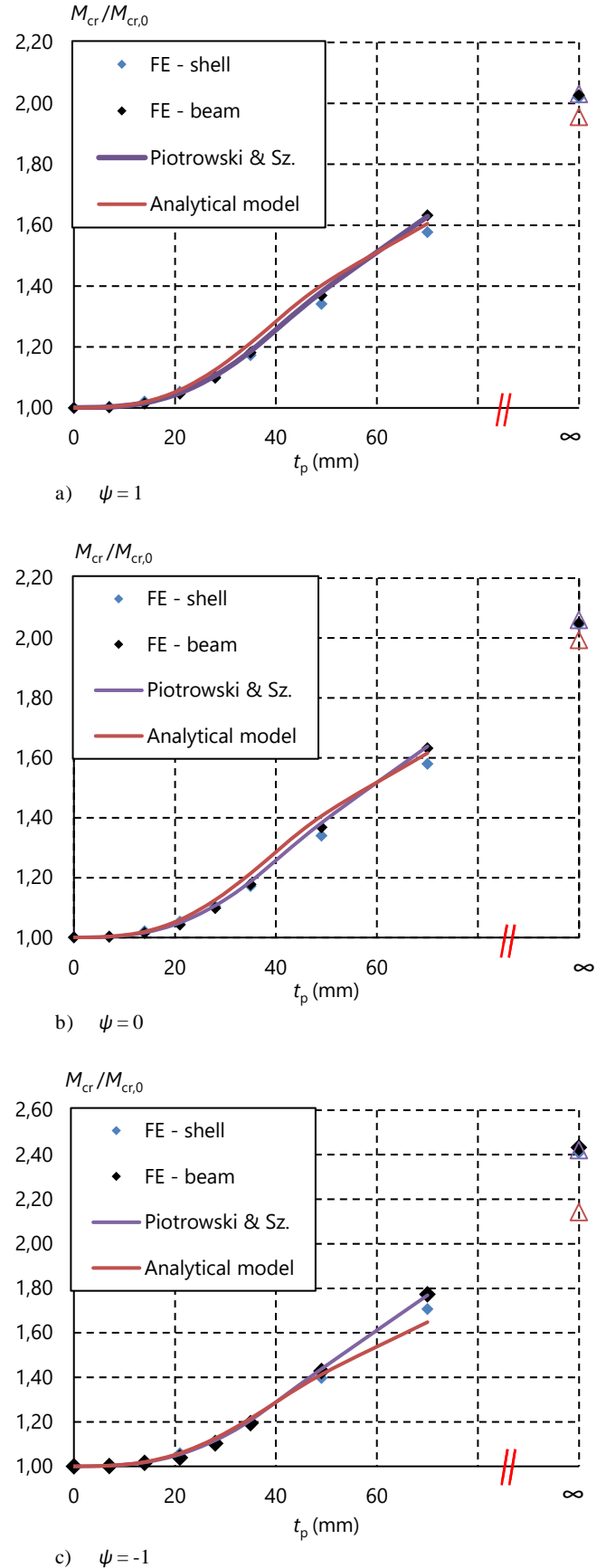


Figure 9 Increase in M_{cr} for P2 under linear moment with warping restraints

When the bending moment distribution is uniform or triangular, all the methods give the same results up to 30mm-thick end plates.

When the warping stiffness increases, the analytical model hardly overestimates (of no more than 5%) the value of the elastic critical bending moment when compared to finite element analyses. When warping is fixed, the analytical model becomes slightly safer, with a difference below 3%.

When $\psi = -1$, the results given by the different methods coincide up to a higher warping restraint. When the warping stiffness increases, the analytical model lays on the safe side but the difference with the shell model is not greater than 10%.

The variation between Piotrowski [6] and the analytical model is only visible when the warping restraints are significant, particularly when $\psi = -1$. When compared to finite element analyses using shell elements, the analytical model always lays on the safe side unlike Piotrowski [6].

Table 6 shows statistical parameters about the calculated values of the critical moment of the beams P1 to P5 (included) presented in Table 5. The ratios between critical bending moments determined with the analytical model and the shell finite element model are presented in Table 6. This table also shows the relations between the critical bending moment increases $M_{cr}/M_{cr,0}$ calculated with the analytical model and FE analyses.

Table 6 Comparison between analytical results and shell finite element analyses

ψ		1	0,5	0	-0,5	-1
$\frac{M_{cr}}{M_{cr,0}}_{an}$	M	1,006	1,006	1,007	1,005	0,976
$\frac{M_{cr}}{M_{cr,0}}_{FEA}$	SD	0,0201	0,0201	0,0206	0,0178	0,0352
$\frac{M_{cr,an}}{M_{cr,FEA}}$	M	1,017	1,021	1,043	1,069	0,954
	SD	0,0207	0,0206	0,0220	0,0275	0,0345

Where:

- M: Mean
- SD: Standard deviation

The elastic critical bending moment evolution estimated by the analytical model is very close to the results of shell finite element analyses. The difference is lower than 1% except when $\psi = -1$ where it increases up to 2,4%.

Critical bending moments determined with the analytical model and FEA are slightly different, but their difference is less than 7%, particularly when ψ is between -0,5 and 0. It matches with the values of ψ for which the analytical and numerical models show small differences in the value of C_1 for free warping (see Figure 8).

The displacement and rotation fields could still be refined with a third term to improve results. However, it would lead to heavier analytical developments, leading to a very cumbersome expression for the equivalent uniform moment factor C_1 . In addition, 7% is still an admissible difference given that the resistant bending moment determined according to Eurocode 3 Part 1-1 [7], $M_{b,Rd}$, depends not only on the critical bending moment but also on many other parameters.

5 Conclusions

The energy method has been used to derive an analytical expression

of the elastic critical bending moment of a beam with uniform doubly symmetric I-section and warping restraints at both ends, subjected to a uniform or linear bending moment distribution. The analytical expression is the same as the French National Annex to Eurocode 3 Part 1-1 [2], but the warping coefficient k_w and the equivalent uniform moment factor C_1 have distinct expressions.

The warping coefficient k_w has been shown to remain unchanged whatever the bending moment distribution and only depends upon the warping stiffness of the beam EI_w , its length L and the stiffness of the warping restraints c_w . Furthermore, the equivalent uniform moment factor C_1 , only based on the bending moment distribution in the French NA [2], is found to also be depending on the stiffness c_w of the warping restraints.

Results from the analytical model were shown to be in good agreement with finite element analyses computed with the software packages ANSYS and LTBeamN [1].

The increase in the value of the critical moment is limited if the warping restraints are only obtained by end plates, but the benefit can become significant if different end connections are considered (fixed column bases, beam-to-beam or beam-to-column joints...).

Future work may include the derivation of analytical expressions to determine the warping stiffness corresponding to realistic support conditions. In addition, the bending moment distribution may be extended to generalise the expression of the elastic critical moment.

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