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Daniel Bermudez, Raju Roychowdhury

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Towards a T-dual Emergent Gravity

dedicated to the city of S \tilde{A} O PAULO, then, now and forever on the 467th anniversary

Daniel Bermudez \heartsuit and Raju Roychowdhury \clubsuit, \diamond

\clubsuit *Institute of Mathematics and Computation
Universidade Federal de Itajuba
Av. BPS 1303, 37500-903, Pinheirinho, Itajuba - MG, Brazil*

\diamond *Institute of Physics, Universidade de S \tilde{a} o Paulo,
C. Postal 66318, 05314-970, S \tilde{a} o Paulo-SP, Brazil
raju.roychowdhury@gmail.com, raju@unifei.edu.br*

\heartsuit *Mathematical Institute University of Bonn,
53012, Bonn, Germany
danielbermudez@uni-bonn.de*

Abstract

Darboux theorem in symplectic geometry is the crux of emergent gravity in which the gravitational metric emerges from a noncommutative $U(1)$ -theory. Topological T-duality, on the other hand, is a relation between two a priori different backgrounds (with different geometries, different fluxes and even topologically distinct manifolds) which nevertheless behave identically from a physical point of view. For us these backgrounds are principal torus bundles on the same base manifold. In this article we review how these theories can be naturally understood in the light of generalized geometry. Generalized geometry provides an unifying framework for such a systematic approach and gives rise to the *group of Courant automorphism* $\text{Diff}(M) \ltimes \Omega_{\text{closed}}^2(M)$ for the $TM \oplus T^*M$ bundle. Here we propose a novel geometric construction for the T-dual of an emergent gravity theory implemented between the \mathbb{T} -bundles and this duality is realised using the *Gualtieri-Cavalcanti map* that establishes an isomorphism between *Courant algebroids*. In the case of flat spacetime we obtain that, under mild assumptions, the T-dual of emergent gravity is again an emergent theory of gravity. In the general case we obtained formulas for the T-dual of an emergent metric in a \mathbb{T}^2 -fibration, however due to the appearance of H -flux after T-dualizing, the theory thus obtained can no longer be considered in the usual framework of emergent gravity. This motivates the study of emergent gravity with non trivial H -flux.

1 Introduction

It is true that gauge theories on noncommutative (NC) spacetime are way more comprehensible than gravity in a similar situation. Seiberg Witten map [1] enables us to rewrite a gauge theory on a NC flat space as an interacting nonlinear gauge theory on a general curved background which is produced by the gauge field itself. For a gauge field with a nontrivial curvature, even if we start with a NC space that is flat, we end up with a background that has nonzero curvature. This phenomenon can be ascribed to the emergence of a geometric structure from gauge theory on NC space and thus was called emergent gravity [2]. For a detailed account on emergent gravity readers may consult the reviews written by Yang [3] and [4] for the quantization program.

More recently one research group in Japan have published series of works [5] where they have described D-branes in the generalized geometric framework identifying them in a static gauge including fluctuations with a leaf of foliation generated by the Dirac structure of a generalized tangent bundle. The scalar fields and the gauge fields on the D-brane were treated on equal footing as a generalized connection. Meanwhile, Jurco et. al in [6] have postulated that the Seiberg-Witten (SW) equivalence between the commutative and the semi-classically NC DBI actions have its very root ingrained in the generalized geometry of D-branes. Specifically, if we consider D-brane as a symplectic leaf of the Poisson structure, describing the noncommutativity, the SW map has a natural interpretation in terms of the corresponding Dirac structure. Thus generalized geometry provides a natural framework for the study of NC gauge theories. It has been argued by Yang et. al. [2, 3] why electromagnetism in NC spacetime should be a theory of gravity, showing that NC electromagnetism [7] can be realized through the Darboux theorem [8] in symplectic geometry, which is the crux of emergent gravity, relating the deformation of a symplectic structure with diffeomorphism symmetry. This observation lead us to speculate that the mathematical structure of emergent gravity which is intimately related to NC electromagnetism can be better understood equipped with the machineries of generalized geometry.

Generalized geometry was introduced by Hitchin [9] in the early 2000s and later developed by Gualtieri and Cavalcanti [10] as a framework for studying geometric structures on exact Courant algebroids, which are essentially the double of the tangent bundle i.e. the direct sum of tangent and cotangent bundles. But there are more structures on them namely the one related to 3-form flux H on the underlying manifold M with $dH = 0$. This H field in generalized geometry has its incarnation in Type II strings where the bosonic field called the NS flux is also a closed 3-form. The exact Courant algebroid provides the mathematical setting for the description of the NS sector of the type II supergravity backgrounds in terms of generalised geometry. The group $O(n, n; \mathbb{R})$ acts on such backgrounds with a Lorentzian metric g and Kalb-Ramond field $B \in \Gamma[\Lambda^2 T^*]$ on a n -dimensional manifold M (where T, T^* are the tangent and cotangent bundles of M , respectively) and this action naturally arises from string theory: backgrounds related by the action of the discrete subgroup $O(n, n; \mathbb{Z})$ define physically equivalent string theories and this equivalence is known as T -duality. The $O(n, n; \mathbb{R})$ -action on (g, B) is best described through the introduction of a generalised metric \mathcal{G} . Gualtieri in his thesis [10] characterized the solutions to Type II string theory with extended $\mathcal{N} = 2$ supersymmetry as generalized Kahler manifolds. Later on, Gualtieri along with Cavalcanti [11] rephrased the T duality of Type II strings as an isomorphism of Courant algebroids.

This project started with an intuition that is encapsulated into a block diagram (see figure 1). Let us begin with the following four independent observations. First, in [6] Jurco, Schupp and Vysoky argued how the equivalence (SW map) between commutative and semi-classically noncommutative DBI actions, which is at the heart of emergent gravity, can be encoded in the generalized geometric framework. Secondly, from the celebrated works of Gualtieri and Cavalcanti [10, 11] the relation between generalized geometry and T duality was made clear using the isomorphism between Courant algebroids and thus transporting invariant geometric structures between the T dual pairs. Thirdly, in recent past, Yang in [12] have discovered mirror symmetry in six space time dimensions in an emergent gravity context for Calabi-Yau spaces using the Hodge theory of deformation of the symplectic and the dual symplectic structures. And finally, there is the classic 25 year old result [13] of Strominger, Yau and Zaslow setting up an equivalence between mirror transformation and T -duality on toroidal 3-cycles for a generic Calabi Yau manifold X and its mirror Y . All these four facts that has been drawn in blocks as part of the following diagram conspires to let us think that there must be some kind of bridge between emergent gravity and T -duality, in other words, we suspected of a T dual avatar of emergent gravity to exist in this quadrilateral framework that encompasses all of the above mentioned observations. In a nutshell we were after finding the right bridge (drawn as dotted line) between the two blocks namely emergent gravity and T -duality in figure 1.

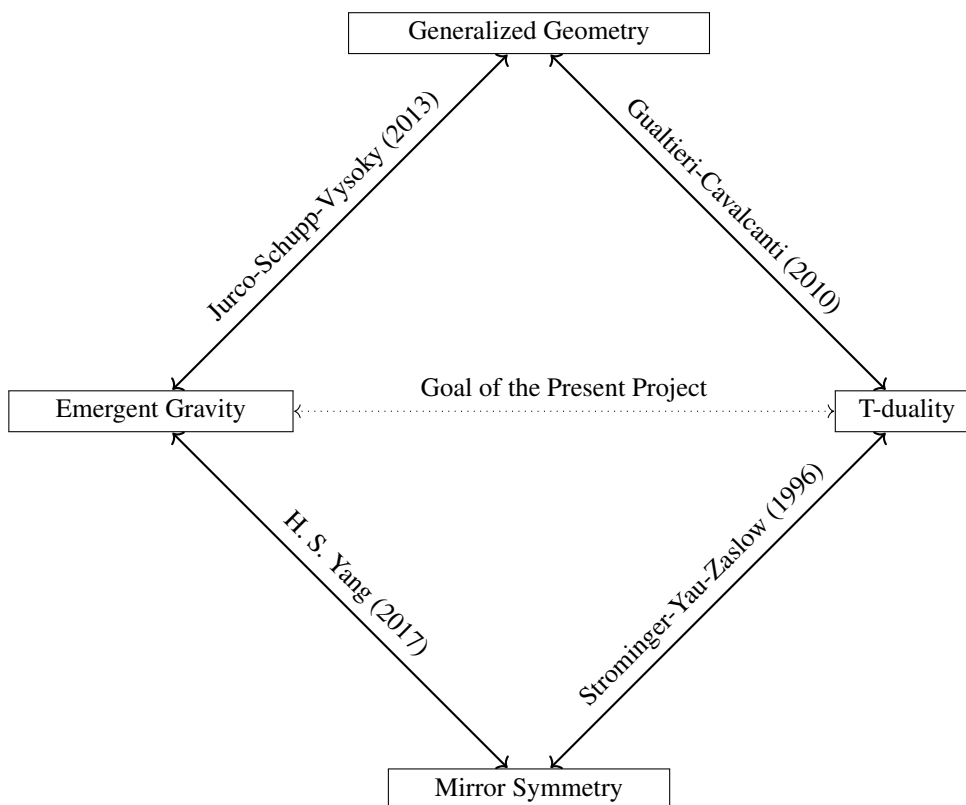


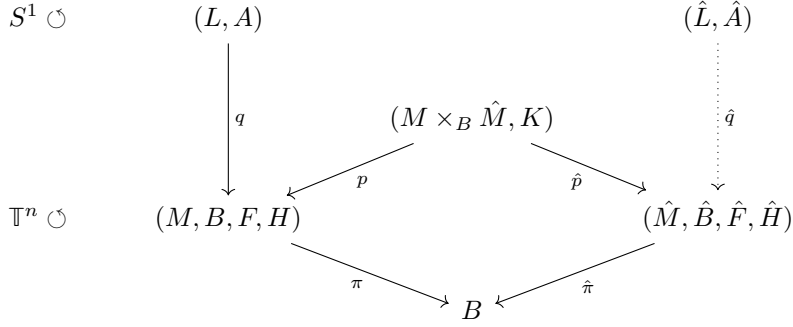
Figure 1: Flowchart for emergent gravity/T duality correspondence

The purpose of the present paper is manifold. We shall consider T-duality in the context of emergent gravity and unravel the generalized geometry origin of emergent gravity. After identifying the generalized metric for \mathbb{T} -invariant emergent gravity we will use the machinery developed by Gualtieri-Cavalcanti (GC) [11] to transport generalized metric and the symplectic structure to the dual torus bundle. Our aim is to provide an explicit formula for the T dual generalized structures that can be obtained via isomorphisms of Courant algebroids. For the 2-torus action that will be considered in this work, we see that T duality changes from symplectic to symplectic whereas for S^1 action [11] T duality relates symplectic geometry on one side to the complex on the other side.

The plan of the present article is as follows. We shall begin by presenting a page long glossary of emergent gravity including a user friendly diagram that will be used for further formulation. In Sec.2 we will briefly discuss the ambient of generalized geometry that leads to the notion of exact Courant algebroids (subsection 2.1) in the context of emergent gravity. We shall work with examples of Courant algebroids that are related to principal bundles. We shall recall few basic definitions and the existence of the *Atiyah sequence* corresponding the principle bundle of emergent gravity in other subsections. Sec.3 will be devoted to *Generalized metrics* as they appear in our theory as a generalization of the tangent bundle of the manifold. In Sec.4 we recall the notion of mathematical T duality as was introduced by Bouwknegt, Evslin, Mathai and Hannabuss [14] as a relation between principal torus bundles and conjecture a plausible *T-dual* candidate for the emergent gravity proposing its implementation via Gualtieri-Cavalcanti map (subsection 4.2). We also present a schematic diagram for topological T duality for torus action here. In subsection 4.3 we shall derive an explicit expression for the T-dual of the symplectic form with symplectic fibers using the notions of transport of invariant geometric structures under GC map. The interconnection between emergent gravity and T-duality will be further emphasized in the first subsection of Section 5, while the rest of the section will be devoted to spell out the main findings of ours for flat spacetimes (with base as a point) and non-flat spacetimes (with arbitrary base) highlighting the fact that emergent gravity can be described by a series of operations, namely B field transform first followed by a θ transform, on the flat metric, while on the T-dual avatar the order of these operations gets swapped as will be clear from the commutative diagram that will appear in this section. Although emergent gravity picture in H-flux background is not quite clear, yet, we shall produce original formulas for the components of the emergent generalized metric and its T-dual counterpart defining the very basis

for the T-dual emergent gravity from a generalized geometric perspective. Finally in Sec.6 we summarise and make some concluding remarks for further developments. Additional details of computations presented in subsection 5.3 can be found in the appendix.

Glossary:



Emergent gravity data:

- M : The spacetime manifold (one that admits a symplectic structure)
- B : symplectic structure on M i.e. a non-degenerate closed 2-form
- L : Line bundle over M – $U(1)$ -theory on M
- A : $U(1)$ -connection of L – Potential of the $U(1)$ -theory on M .
- F : The curvature of the connection L , i.e. $F = dA$ – The electromagnetic field of the $U(1)$ -theory.
- $\theta = B^{-1}$: The Poisson structure associated to B .
- G : The effective Riemmanian metric on M determined by B and F , to be precise $g = 1 + F\theta$. Since G emerges from $U(1)$ gauge fields, we call it *emergent metric*. It is in general not symmetric because $g - g^T = F\theta - \theta F \neq 0$. We will assume in this work $F\theta = \theta F$
- \mathcal{G} : Generalized metric on $TM \oplus T^*M$ determined by the pair (g, B) , where g is a Riemannian metric on M and B is a 2-form (in this case the symplectic form). \mathcal{G} is the endomorphism of $TM \oplus T^*M$ defined by:

$$\mathcal{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

- The generalized metric is completely defined by the subbundle C_+
 $= \text{Ker}(\mathcal{G} - 1) = \text{graph}(g + B : TM \rightarrow T^*M)$.

T-dual emergent gravity data:

- \hat{M} : The dual spacetime of interest, however is also a Torus bundle over B .
- $\hat{\mathcal{G}}$: The transformed generalized metric on $T\hat{M} \oplus T^*\hat{M}$ is given by \hat{C}_+ .
- $\hat{C}_+ = \text{graph}(\hat{g} + \hat{B} : T\hat{M} \rightarrow T^*\hat{M})$ where (\hat{g}, \hat{B}) are given by the Buscher rules.
- (a_i) : \mathfrak{t}^n -valued connection of the Torus bundle $M \rightarrow B$: is a tuple of one forms, it gives a decomposition $T^*M = T^*B \oplus \langle a_1, a_2, \dots, a_n \rangle$.
- H : The 3-form or the H -flux on M , in emergent gravity $H = dB = 0$, although \hat{H} is not necessarily zero.
- K : The 2-form on correspondence space such that $p^*H - \hat{p}^*\hat{H} = dK$, in the $H = 0$ case, $K = \sum p^*a_i \wedge \hat{p}^*\hat{a}_i$.

2 Generalized Geometry and Emergent Gravity

A Riemannian geometry is defined by a pair (M, g) where the metric g encodes all geometric information while a symplectic geometry is defined by a pair (M, B) where the 2-form B captures all. Generalized Geometry was an attempt to merge both of them together in a single package that was first introduced by N. Hitchin [9] in 2002 and further developed by M. Gualtieri and G. R. Cavalcanti [10].

Let M be a smooth real manifold of dimension n and TM its tangent bundle. To motivate the notion of Courant algebroids let us consider the bundle $TM \oplus T^*M$ on M . The bundle $TM \oplus T^*M$ has a canonical $(n; n)$ signature pseudo-metric (a canonical fiberwise non-degenerate bilinear form of signature (n, n))

$$\langle X + \xi, Y + \eta \rangle = \iota_X \eta + \iota_Y \xi = \eta(X) + \xi(Y), \quad (2.1)$$

where $X, Y \in \Gamma(TM)$ are vector fields and $\xi, \eta \in \Omega^1(M)$ are 1-forms.

We also have a natural projection $\pi : E = TM \oplus T^*M \rightarrow TM$ to the first coordinate which is a smooth bundle map called anchor. In the case of a differentiable manifold the space of section is endowed with a derived bracket, the Lie bracket, associated to the de Rham complex. It turns out that there is a derived bracket on the double bundle $TM \oplus T^*M$ which is mapped via the anchor π to the Lie bracket. A skew-symmetric bracket was first defined by Courant in [15], then a nonskew-symmetric version by Dorfman [16]. The two brackets encode the same data and called either Courant or Dorfman brackets. In defining the bracket we shall follow the convention of [17] where the non skew-symmetric bracket is derived in the same manner as the Lie bracket of vector fields utilising actions on the differential forms.

On a smooth manifold M vector fields act on the graded exterior algebra of differential forms via the interior product $X \cdot \varphi = i_X \varphi$ $X \in \Gamma(TM)$, $\varphi \in \Omega^\bullet(M)$. The interior product is a degree -1 operator and we also have the exterior differential d acting on $\Omega^\bullet(M)$ which is of degree +1. The Lie bracket of two vector fields $X; Y$ is then defined as the unique section of TM satisfying

$$\iota_{[X, Y]} \cdot \varphi = [\mathcal{L}_X, \iota_Y] \cdot \varphi = [[\iota_X, d], \iota_Y] \cdot \varphi \quad \forall \varphi \in \Omega^\bullet(M) \quad (2.2)$$

The commutators are meant to be supercommutators of operators

$$[A, B] = A \circ B - (-1)^{|A| \cdot |B|} B \circ A \quad (2.3)$$

where $|A|$ denotes the degree of the operator A .

Now notice that $TM \oplus T^*M$ also acts on forms via the Clifford action

$$(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi \quad (2.4)$$

One can then define the Dorfman bracket of two sections $e_1, e_2 \in \Gamma(TM \oplus T^*M)$ as the unique section satisfying

$$[e_1, e_2] \cdot \varphi = [e_1, [d, e_2]] \cdot \varphi \quad (2.5)$$

Here although the action $TM \oplus T^*M$ has mixed degree, both parts are of odd degree and hence $[d, e_1] = d \circ e_1 + e_1 \circ d$. Writing out the action one finds what is known as Dorfman bracket

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi \quad (2.6)$$

of two sections $(X + \xi)$ and $(Y + \eta)$ on the space of sections of $\Gamma(E)$.

The 4-tuple $(T \oplus T^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ is the first example of a *Courant algebroid*. These objects were axiomatized by Liu, Weinstein and Xu in [18] for the skew-symmetrized version of the bracket.

We will sometimes by abusing notation denote a *courant algebroid* \mathcal{E} simply by E . We say that a Courant algebroid \mathcal{E} is *transitive* if the anchor $\pi : E \rightarrow TM$ is surjective and that it is *exact* if it fits into the short exact sequence of vector bundles

$$0 \longrightarrow T^*M \xrightarrow{\frac{1}{2}\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \quad (2.7)$$

Clearly $TM \oplus T^*M$ is an exact Courant algebroid over M .

2.1 Exact Courant algebroids

Exact Courant algebroids are just a slight generalization of $TM \oplus T^*M$ as an isotropic splitting $s : TM \rightarrow E$ of the sequence (2.7) that renders E isomorphic to $TM \oplus T^*M$ and it becomes the natural pairing. Whenever we talk about a splitting of E , we mean an isotropic splitting of (2.7). Splittings always exist as the inner product has split signature since the image of T^*M is isotropic. Such a splitting $s : TM \rightarrow E$ not only defines an isomorphism $E \cong TM \oplus T^*M$ but also a closed 3-form $H \in \Omega^3(M)$ via

$$H(X, Y, Z) = \langle [s(X), s(Y)], s(Z) \rangle \quad \forall X, Y, Z \in \Gamma(TM). \quad (2.8)$$

Different isotropic splittings of E are globally related by 2-forms $B \in \Omega^2(M)$ which change H by an exact 3-form. Therefore the cohomology class of H is independent of the splitting and it also characterises exact Courant algebroids. The class of H in $H^3(M; \mathbb{R})$ is called the Ševera class [19] of \mathcal{E} . In the split description of $E \cong TM \oplus T^*M$ the bracket is twisted by the 3-form H and takes the form

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H \quad (2.9)$$

We shall now recall two propositions from Gualtieri's D. Phil.thesis [10]; without proving them; that capture the idea of the automorphism group of an exact Courant algebroid. We shall see later that the B-field of emergent gravity satisfy the necessary condition in order to realize e^B as the automorphism in a splitting independent way.

Proposition 2.1.1. *Let $F : E \rightarrow E$ be a vector bundle isomorphism covering the identity on M that is orthogonal with respect to the inner product and preserves the anchor; i.e. $\forall e_1, e_2 \in \Gamma(E)$*

$$1. \langle e_1, e_2 \rangle = \langle F e_1, F e_2 \rangle$$

$$2. \pi(e_1) = \pi \circ F(e_1)$$

Then F is a B -transform for some $B \in \Omega^2(M)$.

On the other hand, such a B-field transform does not necessarily preserve the Dorfman bracket. If $E \cong TM \oplus T^*M$ is equipped with the H -twisted bracket we have

$$\begin{aligned} [e^B(X + \xi), e^B(Y + \eta)]_H &= [X + \xi + \iota_X B, Y + \eta + \iota_Y B]_H \\ &= [X + \xi, Y + \eta]_H + d\iota_X \iota_Y B - \iota_Y d\iota_X B + \iota_X d\iota_Y B \\ &= [X + \xi, Y + \eta]_H + \iota_Y \iota_X B + \iota_{[X, Y]} B \\ &= e^B[X + \xi, Y + \eta]_{H+dB} \end{aligned} \quad (2.10)$$

Therefore e^B is an automorphism of the exact Courant algebroid E if and only if B is closed which is the case in emergent gravity [2, 3, 4]. With this we are ready to describe $Aut(\mathcal{E})$.

Proposition 2.1.2. *The automorphism group $Aut(\mathcal{E})$ of a Courant algebroid is a semidirect product fitting into the following short exact sequence*

$$1 \rightarrow \Omega_{cl}^2(M) \rightarrow Aut(\mathcal{E}) \rightarrow Diff_{[H]}(M) \rightarrow 1 \quad (2.11)$$

Where $Diff_{[H]}(M)$ is the subgroup of diffeomorphisms of M preserving the Ševera class of \mathcal{E} and $\Omega_{cl}^2(M)$ is the space of closed 2-forms on M .

The Lie algebra $Der(\mathcal{E})$ of infinitesimal symmetries is obtained by differentiating one parameter families of automorphisms around the identity. Since these are by definition homotopic to the identity they always preserve the cohomology class $[H]$. Let $\{F_t\} = \{e^{tB} \tilde{\varphi}_t\}$ be such a family with $X \in \Gamma(TM)$ the vector field associated to $\{\varphi_t\} \subset Diff(M)$. Consider a splitting of E with the Dorfman bracket twisted by H . Then

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} e^{tB} \tilde{\varphi}_t(Y + \eta) &= \left. \frac{d}{dt} \right|_{t=0} (\varphi_t)_* Y + (\varphi_{-t})^* \eta + \iota_{(\varphi_t)_*} Y t B \\ &= \mathcal{L}_X(Y + \eta) + \iota_Y B \end{aligned} \quad (2.12)$$

Moreover, the constraint $\varphi_t^* H - H = t \cdot dB$ becomes $\mathcal{L}_X H = dB$ and we have

$$Der(\mathcal{E}) = \{(X, B) \in \Gamma(TM) \times \Omega^2(M) \mid \mathcal{L}_X H = dB\}$$

Consequently we obtain the following splitting independent description of $Der(\mathcal{E})$.

Two more propositions are in order.

Proposition 2.1.3. *The Lie algebra of derivations $\text{Der}(\mathcal{E})$ is an abelian extension of the Lie algebra $\text{Der}(TM)$ fitting into the short exact sequence*

$$0 \rightarrow \Omega_{cl}^2(M) \rightarrow \text{Der}(\mathcal{E}) \rightarrow \text{Der}(TM) \rightarrow 0$$

where $\text{Der}(TM)_{[H]}$ is the space of vector fields corresponding to 1 - parameter groups of diffeomorphisms preserving the cohomology class $[H] \in H^3(M, \mathbb{R})$.

For vector fields $\Gamma(E)$ acts on itself via the adjoint action as a derivation. On the other hand, in this case the map $\Gamma(E) \rightarrow \text{Der}(\mathcal{E})$ is neither injective nor surjective. Instead its image is described by the following proposition.

Proposition 2.1.4. *The adjoint action of $\Gamma(E)$ via the Dorfman bracket fits into the following exact sequence*

$$0 \longrightarrow \Omega_{cl}^1(M) \xrightarrow{\frac{1}{2}\pi^*} \Gamma(E) \xrightarrow{ad} \text{Der}(\mathcal{E}) \xrightarrow{\chi} H^2(M, \mathbb{R}) \longrightarrow 0 \quad (2.13)$$

Proof. Recall that in a certain splitting

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H$$

Clearly, $ad(X + \xi) = 0$ if and only if $X = 0$ and $d\xi = 0$. In the previous proposition we saw that $\text{Der}(\mathcal{E})$ consists of ordered pairs (X, B) acting via the Lie derivative and contraction satisfying $dB = \mathcal{L}_X H$. Therefore we may define

$$\chi(X, B) = [\iota_X H - B] \in H^2(M, \mathbb{R})$$

The map is well defined since H is closed therefore $d(\iota_X H - B) = \mathcal{L}_X H - dB = 0$ and surjective since we may choose $B \in \Omega^2(M)$ arbitrarily. The kernel of χ then consists of (X, B) such that $\iota_X H - B = d\xi$ which act via

$$\begin{aligned} (X, B) \cdot (Y + \eta) &= \mathcal{L}_X(Y + \eta) + \iota_Y B \\ &= [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H \\ &= [X + \xi, Y + \eta] \end{aligned}$$

which proves the exactness of the sequence.

2.2 Atiyah algebroid in emergent gravity

We refer the reader to the page long diagrammatic exposition of emergent gravity that was introduced by Yang [2] by considering the deformation of a symplectic manifold (M, B) where B is a non-degenerate, closed two-form on M . We consider a line bundle L over (M, B) whose connection one-form is denoted by $A = A_\mu(x)dx^\mu$ and the curvature F of the line bundle is a closed two-form, i.e., $dF = 0$ and so locally can be expressed as $F = dA$. The line bundle L over (M, B) admits a local gauge symmetry \mathfrak{B}_L which acts on the connection A as well as the symplectic structure B in the following way :

$$\mathfrak{B}_L : (B, A) \mapsto (B - d\Lambda, A + \Lambda) \quad (2.14)$$

where Λ is an arbitrary one-form on M . The local gauge symmetry \mathfrak{B}_L is known as the Λ -symmetry. This symmetry demands that the curvature $F = dA$ of L appear only as the combination $\mathcal{F} \equiv B + F$ since the two-form \mathcal{F} is gauge invariant under the Λ -symmetry. Since $d\mathcal{F} = 0$, the line bundle L over (M, B) leads to a ‘‘dynamical’’ symplectic manifold (M, \mathcal{F}) if $\det(1 + F\theta) \neq 0$ where $\theta \equiv B^{-1}$ is the Poisson structure associated to B . Therefore the electromagnetic force $F = dA$ manifests itself as the deformation of a symplectic manifold (M, B) . The local gauge symmetry \mathfrak{B}_L induced by the symplectic structure on M , defines a bundle isomorphism $B : TM \rightarrow T^*M$ by $X \mapsto \Lambda = -\iota_X B$ where $X \in \Gamma(TM)$ is an arbitrary vector field such that the Λ transformation can be written as

$$\mathfrak{B}_L : (B, A) \mapsto (B + \mathcal{L}_X B, A - \iota_X B) \quad (2.15)$$

where $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to the vector field X .

First, let us recall some background knowledge on principal bundles and its Atiyah algebroid. Most of these facts can be found in many classic books on the topic, we refer to Kobayashi and Nomizu [20] for basics on principal bundles. In this article are mainly interested in torus bundle, that is principal bundle with structure group \mathbb{T} , however

many definitions and construction holds in general for compact, connected, semisimple structure groups G . This means there is a non-degenerate \mathbb{T} -invariant symmetric bilinear pairing on the Lie algebra \mathfrak{t}_2 of \mathbb{T} . The Lie bracket on \mathfrak{t}_2 is defined as the Lie bracket of left invariant vector fields on \mathbb{T} for sign convention.

Let $\pi : M \rightarrow B$ be a principal \mathbb{T} -bundle. The fundamental vector fields on M are defined as

$$\begin{aligned}\psi : \mathfrak{t}_2 &\rightarrow \Gamma(TM) = \text{Vect}(TM) \\ \psi(x)|_m &= \left. \frac{d}{dt} \right|_{t=0} m \cdot \exp(tx)\end{aligned}\quad (2.16)$$

The map ψ is called the infinitesimal action of \mathfrak{t}_2 . With our sign convention ψ is a Lie algebra homomorphism from \mathfrak{t}_2 to the Lie algebra of vector fields $\text{Vect}(TM)$, i.e. we have

$$\psi([x, y]) = [\psi(x), \psi(y)] = \mathcal{L}_{\psi(x)}\psi(y) \quad (2.17)$$

Fundamental vector fields span the *vertical* subbundle of TM consisting of vectors that are in the kernel of π_* . The action of \mathbb{T} on M lifts naturally to the tangent bundle $TM \rightarrow M$ by differentiation. For $X \in TM$ and $t \in \mathbb{T}$

$$(t \cdot X)|_{mt} = (R_t)_* X = X \quad (2.18)$$

Definition 2.2.1. Let $\pi : M \rightarrow B$ be a principal \mathbb{T} -bundle. Then the *Atiyah algebroid* corresponding to M is the vector bundle $\mathcal{A} = TM/\mathbb{T}$ over $B = M/\mathbb{T}$.

Proposition 2.2.2. The Atiyah algebroid of M is a Lie algebroid over B with surjective anchor induced by $\pi_* : TM \rightarrow TB$. Moreover \mathcal{A} fits into the short exact sequence of Lie algebroids

$$0 \longrightarrow \mathfrak{t}_{2M} \xrightarrow{j} \mathcal{A} \xrightarrow{\pi_*} TB \longrightarrow 0 \quad (2.19)$$

where \mathfrak{t}_{2M} is the vector bundle associated to the adjoint representation of \mathbb{T} . The *sequence* is called the Atiyah sequence corresponding to the principal bundle M . For a proof of this proposition readers may consult [21].

A differential form ω on M is called invariant if $R_t^* \omega = \omega$ for all $t \in \mathbb{T}$. The space of invariant differential forms is denoted by $\Omega^\bullet(M)/\mathbb{T}$. Sections of the Atiyah algebroid are identified with \mathbb{T} -invariant sections of TM and hence the Atiyah sequence induces a filtration of $\Omega^\bullet(M)/\mathbb{T}$

$$\Omega^\bullet(B) = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \dots \subset \mathcal{F}^n = \Omega^\bullet(M)/\mathbb{T} \quad (2.20)$$

where $\mathcal{F}^i = \text{Ann}(\wedge^{i+1} \mathfrak{t}_2)$ and n is the dimension of \mathfrak{t}_2 .

Connections

A connection 1-form on the principal bundle M is an *equivariant* Lie-algebra valued 1-form $A \in \Omega^1(M, \mathfrak{t}_2)$ meaning that

$$R_t^* A = \text{Ad}(t^{-1}) A \quad \forall t \in \mathbb{T} \quad (2.21)$$

where Ad is the adjoint action of \mathbb{T} on the Lie algebra, such that

$$\iota_{\psi(x)} A = x \quad \forall x \in \mathfrak{t}_2 \quad (2.22)$$

By equivariance the connection 1-form descends to a right splitting of the Atiyah sequence. In particular $A(j(x)) = x$ holds $\forall x \in \mathfrak{t}_2$ as well. A connection on M can also be thought of as a choice of \mathbb{T} -invariant *horizontal distribution* H of TM . More precisely for all $t \in \mathbb{T}$ and $m \in M$

$$(R_t)_* H_m = H_{tm} \quad (2.23)$$

and $\forall X \in \Gamma(TB)$ there is a unique \mathbb{T} -invariant $X^H \in \Gamma(H)$ such that

$$\iota_{X^H} A = 0 \text{ and } \pi_* X^H = X \quad (2.24)$$

This viewpoint gives the right splitting of the Atiyah sequence via $X \mapsto X^H$ corresponding to the left splitting given by A .

Given a connection $A \in \Omega^1(M, \mathfrak{t}_2)$ the Atiyah sequence splits as $\mathcal{A} \cong \mathfrak{t}_{2M} \oplus TB$. Then we can identify sections of \mathcal{A} with \mathbb{T} -invariant sections of TM which can be written as

$$\begin{aligned} TB \oplus \mathfrak{t}_{2M} &\rightarrow TM \\ X + s &\mapsto X^H + j(s) = X^H + s \end{aligned} \quad (2.25)$$

where X^H is the horizontal lift and j is the map from the Atiyah sequence. We omit j later, although one has to be careful with signs.

A connection form A on M turns the filtration (2.20) of invariant differential forms into a decomposition. At degree k it is given by

$$\Omega^k(M)/\mathbb{T} = \bigoplus_{i=0}^k \Omega^i(B, \wedge^{k-i} \mathfrak{t}_2) \quad (2.26)$$

Therefore, if $A = A^i e_i$ in some basis $\{e_i\}$ of \mathfrak{t}_2 , any invariant differential 2-form B which is a symplectic form can be written as

$$B = B^0 + B_i^1 A^i + \frac{1}{2} B_{ij}^2 A^{ij} \quad (2.27)$$

where $A^{ij} = A^i \wedge A^j$ and B^0, B_i^1, B_{ij}^2 are basic forms of degree 2, 1 and 0 respectively i.e. pullbacks of forms on base manifold B via π .

2.3 Reduction of the Courant algebroid in emergent gravity

In emergent gravity we start with an exact Courant algebroid E (defined already in the beginning of this section) over the manifold M and in the present article we consider a Lie group \mathbb{T} acting on M via diffeomorphism on the right. We have also assumed M to be a principal bundle with compact connected structure group \mathbb{T} so that the reduced space $M/\mathbb{T} = B$ is a smooth manifold. We want to consider an action of \mathbb{T} on the total space of $E \cong TM \oplus T^*M$ via automorphisms of E and reduce it to a new non-exact Courant algebroid on the base B . Coming from the action of \mathbb{T} we have Lie-algebra homomorphism (2.16) defining the infinitesimal action of \mathfrak{t}_2 on $\Gamma(TM)$ via the adjoint action (Lie bracket) of vector fields.

Definition 2.3.1. A lifted action of \mathbb{T} on E is a right action of \mathbb{T} on E via automorphisms covering the action of \mathbb{T} on M . A lifted infinitesimal action of \mathfrak{t}_2 on E is a Lie algebra homomorphism $\alpha : \mathfrak{t}_2 \rightarrow \text{Der}(E)$ covering the infinitesimal action ψ of \mathfrak{t}_2 on TM .

Given a lifted action of \mathbb{T} on E by differentiation we obtain a lifted infinitesimal action of \mathfrak{t}_2 on E . Conversely if a lifted infinitesimal action is obtained via differentiating a lifted action of \mathbb{T} we say that the lifted infinitesimal action integrates to an action of \mathbb{T} on E . First, we want to see whether under a group action the invariant sections of an exact Courant algebroid could become a Courant algebroid itself.

Let $\sigma : M \rightarrow B$ be a principal \mathbb{T} -bundle and $(E, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \pi)$ an exact Courant algebroid on M so that \mathbb{T} acts via diffeomorphisms preserving the Ševera class of E . As \mathbb{T} is compact by averaging we may chose a splitting $E \cong TM \oplus T^*M$ such that the bracket is twisted by a \mathbb{T} -invariant 3-form $H \in \Omega^3(B)/\mathbb{T}$.

Then there is a natural extended action of \mathbb{T} on $TM \oplus T^*M$ via

$$\varphi(X + \xi) = \begin{pmatrix} \varphi_* & 0 \\ 0 & (\varphi^{-1})^* \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \varphi_* X + (\varphi^{-1})^* \xi \quad (2.28)$$

By Proposition 2.1.2 and eq.(2.11), this action is an automorphism of the H -twisted Courant algebroid if and only if $\varphi^* H = H$ which is satisfied as we chose H to be \mathbb{T} -invariant. Consequently if M is a principal \mathbb{T} -bundle, the group action naturally lifts to an action on E via Courant automorphisms.

Now we can define a new Courant algebroid over the base $B = M/\mathbb{T}$ of M . Firstly, E/\mathbb{T} is a vector bundle over B as the action is free and proper on M and acts via automorphisms on E . The sections of E/\mathbb{T} naturally identify with the \mathbb{T} -invariant sections of E . These sections are closed under the Courant bracket and their inner product is a \mathbb{T} -invariant function on M . Hence $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ descend to well defined operations on $E/\mathbb{T} \rightarrow M/\mathbb{T}$. Finally, to define the anchor for our new Courant algebroid notice that the anchor π of E sends \mathbb{T} -invariant sections of E to \mathbb{T} -invariant sections of TM which project to sections of $T(M/\mathbb{T})$. Therefore we can use π as the anchor for E/\mathbb{T}

$$\pi^\mathbb{T} : E/\mathbb{T} \rightarrow TM/\mathbb{T} \rightarrow T(M/\mathbb{T}) \cong TB \quad (2.29)$$

It is easy to check that $(B, E/\mathbb{T}, \pi^\mathbb{T}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ satisfies the axioms of a Courant algebroid over B . The reduced Courant algebroid is not exact, as the rank of E/\mathbb{T} is too large but still transitive. Moreover, considering the \mathbb{T} -invariant splitting $E \cong TM \oplus T^*M$ it is clear that E/\mathbb{T} becomes a Lie algebroid over B and $(TM/\mathbb{T})^* \cong T^*M/\mathbb{T}$. Hence E/\mathbb{T} fits into the short exact sequence

$$0 \longrightarrow (TM/\mathbb{T})^* \xrightarrow{\frac{1}{2}\pi^*} E/\mathbb{T} \xrightarrow{\pi} TM/\mathbb{T} \longrightarrow 0 \quad (2.30)$$

Therefore, the resulting Courant algebroid is the double of the Lie algebroid TM/\mathbb{T} . This is what can be called simple reduction for emergent gravity. This construction easily generalizes to any Courant algebroid E over a principal \mathbb{T} -bundle $M \rightarrow B$. Whenever the action of \mathbb{T} lifts to E via Courant algebroid automorphisms one can define a new Courant algebroid E/\mathbb{T} over the base B . We also call this the simple reduction of E in emergent gravity.

3 Generalized metric and related symmetries

In this section we will introduce the generalization of Riemannian metrics on exact Courant algebroids and see how they behave under reduction following the calculations and definitions of [10]. The term generalized metric was used for the first time by Hitchin in [22].

Definition 3.0.1 A generalized metric on E is a smooth self-adjoint, orthogonal bundle automorphism $\mathcal{G} : E \rightarrow E$ which is positive definite in the sense that $\langle \mathcal{G}e, e \rangle > 0$ for all non-zero sections $e \in \Gamma(E)$. Orthogonality implies that $\mathcal{G}\mathcal{G}^* = Id$, and together with the self-adjoint property we find that $\mathcal{G}^2 = Id$. Therefore, a generalized metric also defines a decomposition of E into its +1 and -1 eigenspaces which we denote by

$$E_- = \ker(Id + \mathcal{G}) \text{ and } E_+ = \ker(Id - \mathcal{G}) \quad (3.1)$$

As \mathcal{G} is positive definite the restriction of the inner product $\langle \cdot, \cdot \rangle$ to E_- is negative definite and to E_+ is positive definite. Denote the projections to the subbundles E_\pm as

$$\Pi_\pm = \frac{1}{2}(Id \mp \mathcal{G}) : E \rightarrow E_\pm \quad (3.2)$$

From this description it is clear that E_+ and E_- are orthogonal with respect to the inner product.

Thus, if the inner product on a Courant algebroid has signature (p, q) then a generalized metric is equivalent to a decomposition of E into two orthogonal subbundles $E = E_+ \oplus E_-$ of rank p and q respectively, such that the restriction of the inner product to E_+ (E_-) is positive (negative) definite. Clearly, defining one of the bundles determines the other as $E_+ = E_-^\perp$. One can then recover the bundle morphism \mathcal{G} by defining it as the identity on E_+ and minus the identity on E_- .

For an exact Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ on a manifold M of dimension n the inner product has signature (n, n) . Therefore a generalized metric is equivalent to defining a rank n negative definite subbundle $E_- \subset E$ and we have the decomposition $E = E_+ \oplus E_-$ for $E_+ = E_-^\perp$. Generalized metrics can also be described from the point of view of the underlying manifold utilising the bundle map $\lambda = (\pi_-)^{-1} : TM \rightarrow E$.¹

Thus we have the following proposition.

Proposition 3.0.2 A generalized metric E_- on an exact Courant algebroid E over the manifold M is equivalent to a Riemannian metric g on M and an isotropic splitting $E \cong TM \oplus T^*M$ such that E_- and E_+ are of the form

$$\begin{aligned} E_- &= \{X - gX \mid X \in TM\} \\ E_+ &= \{X + gX \mid X \in TM\} \end{aligned} \quad (3.3)$$

Note that while a usual metric is a reduction of the frame bundle from $GL(n)$ to the maximal compact subgroup $O(n)$, a generalized metric is a reduction from $O(n; n)$ to $O(n) \times O(n)$.

The symmetry group of the form (2.1) is the orthogonal group

$$O(TM \oplus T^*M) = \{A \in GL(TM \oplus T^*M) \mid \langle A \cdot, A \cdot \rangle = \langle \cdot, \cdot \rangle\} \quad (3.4)$$

¹Given a generalized metric on a transitive Courant algebroid, the anchor restricted to the subbundles E_- and E_+ $\pi_\pm = \pi|_{E_\pm} : E_\pm \rightarrow TM$ is surjective.

Since the bilinear form has signature (n, n) , we have $O(TM \oplus T^*M) \cong O(n, n)$ The Lie algebra

$$o(TM \oplus T^*M) = \{Q \in M(TM \oplus T^*M) \mid \langle Q \cdot, \cdot \rangle + \langle \cdot, Q \cdot \rangle = 0\} \quad (3.5)$$

that consists of matrices of the form

$$Q = \begin{pmatrix} A & \theta \\ B & -A^T \end{pmatrix} \quad (3.6)$$

where

$$\begin{aligned} A : \Gamma(TM) &\rightarrow \Gamma(TM), & A^T : \Gamma(T^*M) &\rightarrow \Gamma(T^*M) \\ \theta : \Gamma(T^*M) &\rightarrow \Gamma(TM), & B : \Gamma(TM) &\rightarrow \Gamma(T^*M) \end{aligned} \quad (3.7)$$

satisfy $\theta^T = -\theta$ and $B^T = -B$. Hence we can think of B as a 2-form $B \in \Gamma(\wedge^2 T^*M) = \Omega^2(M)$ by $\iota_X B = B(X)$ and similarly θ as a bivector $\theta \in \Gamma(\wedge^2 TM)$. We thus see that generalized geometry, by its very nature incorporates 2-forms, i.e. B-fields. The finite transformations corresponding to B and θ are given by

$$\begin{aligned} e^B &\equiv \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, & e^B(X + \xi) &= X + \xi + \iota_X B \\ e^\theta &\equiv \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, & e^\theta(X + \xi) &= X + \iota_\xi \theta + \xi \end{aligned} \quad (3.8)$$

We refer to e^B as a B-field transform. The Courant bracket on $\Gamma(TM \oplus T^*M)$, which plays a similar role in generalized geometry as the Lie bracket on $\Gamma(TM)$, is defined in [15].

By the proposition stated above a generalized metric on E is equivalent to the pair (g, H) where g is the Riemannian metric on M and H is the representative of the Ševera class [19] of E defined by the splitting.

Two-forms act transitively on isotropic splittings of E via the B -transform. Therefore, in an arbitrary splitting $E \cong TM \oplus T^*M$ the two subbundles defined by a generalized metric have the following form:

$$\begin{aligned} E_+ &= \{X + g(X) + B(X) \mid X \in TM\} \\ E_- &= \{X - g(X) + B(X) \mid X \in TM\} \end{aligned} \quad (3.9)$$

Using this formulation, one can reconstruct $\tau : \Gamma(E) \rightarrow \Gamma(E)$ as a $C^\infty(M)$ -linear map of sections, such that $\tau^2 = 1$. For, elements $e_1, e_2 \in \Gamma(E)$, we have

$$(e_1, e_2)_\tau := \langle \tau(e_1), e_2 \rangle = \langle e_1, \tau(e_2) \rangle \quad (3.10)$$

such that $(\cdot, \cdot)_\tau$ is symmetric and defines a positive definite fiberwise metric, referred to as generalized metric on E . Also, since $\tau^2 = 1$, it is orthogonal and thus $\tau \in O(n, n)$. Moreover, we get two eigenbundles E_+ and E_- , corresponding to +1 and -1 eigenvalues of τ . Using g and B we can construct

$$\tau(X + \xi) = (g - Bg^{-1}B)(X) - g^{-1}B(X) + Bg^{-1}(\xi) + g^{-1}(\xi) \quad (3.11)$$

for all $(X + \xi) \in \Gamma(E)$. In the block matrix form,

$$\tau \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \quad (3.12)$$

The corresponding fiberwise metric $(\cdot, \cdot)_\tau$ can then be written in the block matrix form

$$(X + \xi, Y + \eta)_\tau = \begin{pmatrix} X \\ \xi \end{pmatrix}^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix} \quad (3.13)$$

The block matrix in formula (3.13) can be written as a product of simpler matrices. Namely,

$$\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \quad (3.14)$$

The generalized metric \mathcal{G} (eq.3.13)

$$\mathcal{G}^B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

is the B-transform of

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \quad (3.15)$$

i.e.

$$\mathcal{G}^B = e^B G e^{-B}. \quad (3.16)$$

It must be noted that the 2-form B does not have to be closed, and this will remain true all throughout the present work. For the sake of calculation we assume that B is globally defined, and $H = dB$ globally corresponding to a trivial integral cohomology class $[H]$. In emergent gravity we consider only those models with trivial H -flux. The case of the non-trivial H -flux will appear after T dualizing the original theory that is the content for the next section.

4 A plausible T-dual

T duality is a symmetry relating two *a priori* different string backgrounds within the theory, which nevertheless behave identically from a physical point of view. The artifact of T-duality is that Type IIA and IIB theories are really two different manifestations of the same theory with an interchange of “winding” and “momentum” modes and replacing the tori in the spacetime by their dual in the target space. The *local* relationship between T-dual theories, or to be precise, the local transformation rules of the low energy effective fields under T-duality can be understood under the framework of the Buscher rules [23]. On the other hand, the study of the global aspects of T-duality, such as topological changes of the manifolds associated to T-dual theories, began since [24] and were discussed at length in [14] which attempt to capture the basic aspects of the *algebraic and topological* content in T-duality.

In the low energy limit of type II strings, the Bosonic field contents are a metric g , closed 3-form H and the dilaton φ that satisfy modified Einstein equations. Surprisingly these set of equations possess a symmetry, namely T-duality that is not found in the ordinary Einstein equations and this symmetry relates spaces M, \hat{M} which are torus bundles over a common base space B and can be characterised by an interchange of the Chern classes between the torus bundles with topological data associated to the closed 3-form flux H . As is now quite well-known, it is possible to give a geometric meaning to T-duality using the language of generalised geometry [11]. Using this point of view, T-duality can be seen as an isomorphism of twisted exact Courant algebroids. To be precise given $M \rightarrow B$ a principal \mathbb{T} -bundle and the twisted Courant algebroid defined on double bundle $TM \oplus T^*M$ by the 3-form $H \in \Omega^3(M)$, we say there is a T-dual theory if there exist principal torus bundle $\hat{M} \rightarrow B$ and $\hat{H} \in \Omega^3(\hat{M})$, known as the T -dual space and T-dual flux respectively, such that $(TM \oplus T^*M, H) \cong (T\hat{M} \oplus T^*\hat{M}, \hat{H})$. Leaving aside the dilaton φ , the field content (g, H) defines a generalised metric on the Courant algebroid $TM \oplus T^*M$, then it is possible to understand the T-duality of the type II string equations as an isomorphism of generalised metrics. This is what we are going to use precisely for the construction of our T-dual emergent gravity model.

4.1 Existence and Obstruction of T-dual Theories

A priori the two T -dual bundles $M \rightarrow B$ and $\hat{M} \rightarrow B$ are completely unrelated, thus in order to compare them one needs to introduce the correspondence space

$$M \times \hat{M} = \{(m, m') \in M \times M' : \pi(m) = \pi'(m')\} \quad (4.1)$$

i.e. the fiber product with respect to the canonical bundle projections. The two bundles fit into a diagram

$$\begin{array}{ccc} & (M \times_B \hat{M}, p^*H - \hat{p}^*\hat{H}) & \\ & \swarrow p \qquad \searrow \hat{p} & \\ (M, H) & & (\hat{M}, \hat{H}) \\ & \swarrow \pi \qquad \searrow \hat{\pi} & \\ & B & \end{array}$$

Notice that the correspondence space is a \mathbb{T} -bundle with base M with action induced from the \mathbb{T} -action on \hat{M} , similarly it is a \mathbb{T} -bundle over \hat{M} with the induced \mathbb{T} -action on M . Therefore there is an action of $\mathbb{T} \times \mathbb{T}$ on the correspondence space.

Definition 4.1.1 Let M, \hat{M} be two principal \mathbb{T} -bundles over a common base B and let $H \in \Omega^3(M), \hat{H} \in \Omega^3(\hat{M})$ be integral closed forms. We say M and \hat{M} are T-dual if $p^*H - \hat{p}^*\hat{H} = dK$, for some $\mathbb{T} \times \mathbb{T}$ -invariant form $K \in \Omega^2(M \times_B \hat{M})$ which is non degenerate as a form restricted to the subbundle $\mathfrak{t}_M \otimes \mathfrak{t}_{\hat{M}}$.

The obstruction to the existence of T -dual is well understood: let M be a principal torus bundle, then M is T -dualizable if and only if there exists a closed integral \mathfrak{t}^* -valued² 2-form $G \in \Omega^2(B, \mathfrak{t})$ on B , such that the pair (H, G) satisfies

$$dH = 0, \quad \iota_X H = \pi^* G(M), \quad (4.2)$$

for all $X \in \Gamma(\mathfrak{t}_M)$. Some comments are in place: On one hand in the whole T -dual discussion one cares only for phenomena up to (co)homology, thus one is interested in relations up to (co)homology, moreover since every one form is homotopic to an invariant one then we can restrict our whole discussion to \mathbb{T} -invariant objects, as presented in section 2.2. On the other hand, T -duals are not unique, however up to torsion fix there is a unique, up to isomorphism, T -dual \mathbb{T} -bundle. To find the torsion free T -dual to a given T -dualizable bundle M one argues as follows: Since the \mathfrak{t} -valued 2-form G is integral it defines a class in $H^2(M, \mathbb{Z}^n)$ which, by the long exact sequence of cohomology induced by the exponential map, defines in turn a class in $H^1(B, \mathbb{T})$, i.e. a \mathbb{T} -bundle $\hat{M} \xrightarrow{\hat{\pi}} B$. If we demand for the bundle to be torsion free it follows that \hat{M} is unique up to isomorphism. Consider connection one forms $a \in \Omega^1(M, \mathfrak{t})$ and $\hat{a} \in \Omega^1(\hat{M}, \mathfrak{t}^*)$ such that

$$da = \pi^* G, \quad d\hat{a} = \hat{\pi}^* G, \quad (4.3)$$

this always can be done, see for instance [25] for the case of circle bundles. Now define

$$h := a \wedge d\hat{a} - H, \quad (4.4)$$

since H, a and \hat{a} are assumed to be invariant and $\iota_X h = d\hat{a}$, it follows that h is basic, i.e. the pullback of a form in B , abusing notation we will also denote this form by $h \in \Omega^3(B)$. At last the T -dual NS flux can be defined as

$$\hat{H} = da \wedge \hat{a} - h, \quad (4.5)$$

where again we are abusing notation and using da is basic to consider it as a form in the base B . It is clear H is a closed invariant form. Moreover the couple (\hat{M}, \hat{H}) is a T -dual bundle to (M, H) , indeed

$$p^*H - \hat{p}^*\hat{H} = a \wedge d\hat{a} - da \wedge \hat{a} = -d(a \wedge \hat{a}), \quad (4.6)$$

and restricted to the fibers $\mathfrak{t}_M \otimes \mathfrak{t}_{\hat{M}}$ the form $a \wedge \hat{a}$ is just the canonical pairing between \mathfrak{t} and \mathfrak{t}^* therefore non-degenerate. In the case of emergent gravity we are interested in the case $H = db$ for a closed B -field b , thus in the case of interest we will deal with trivial flux $H = 0$. We will come back to this case after developing how to understand T -duality in the context of Courant algebroids and generalized geometry.

4.2 T-duality as an isomorphism of Courant algebroids

In the work by Calvancanti and Gualteri [11], T-duality is exposed in the context of generalized geometry as an isomorphism of Courant algebroids associated to the twisted generalized tangent bundles of (M, H) and (\hat{M}, \hat{H}) respectively. This description allows to transport generalized geometry from one T -dual bundle to another.

We begin with the observation that upon choosing connections $a \in \Omega^1(M, \mathfrak{t})$ the bundles $(TM \oplus T^*M)/\mathbb{T}$ and $(T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ decompose as

$$(TM \oplus T^*M)/\mathbb{T} \cong TB \oplus \langle \partial \rangle \oplus T^*B \oplus \langle a \rangle, \quad (4.7)$$

where $\partial, \hat{\partial}$ are the vertical vector fields associated to the connection forms a, \hat{a} respectively. In the case of a n -torus \mathbb{T}^n the connection forms can be actually described by n one forms $a = (a^1, \dots, a^n)$ and the vector fields $\partial = (\partial_1, \dots, \partial_n)$ are the dual vector fields satisfying $a^i(\partial_j) = \delta_j^i$. Analogously choosing a connection on the T -dual bundle $\hat{a} \in \Omega^1(\hat{M}, \mathfrak{t}^*)$ yield a decomposition

$$(T\hat{M} \oplus T^*\hat{M})/\mathbb{T} \cong TB \oplus \langle \hat{\partial} \rangle \oplus T^*B \oplus \langle \hat{a} \rangle. \quad (4.8)$$

² \mathfrak{t}^* denotes the dual of \mathfrak{t} , usually denoted by $\hat{\mathfrak{t}}$ which we avoid since we will be using the notation $\hat{\cdot}$ for T -dual structures later.

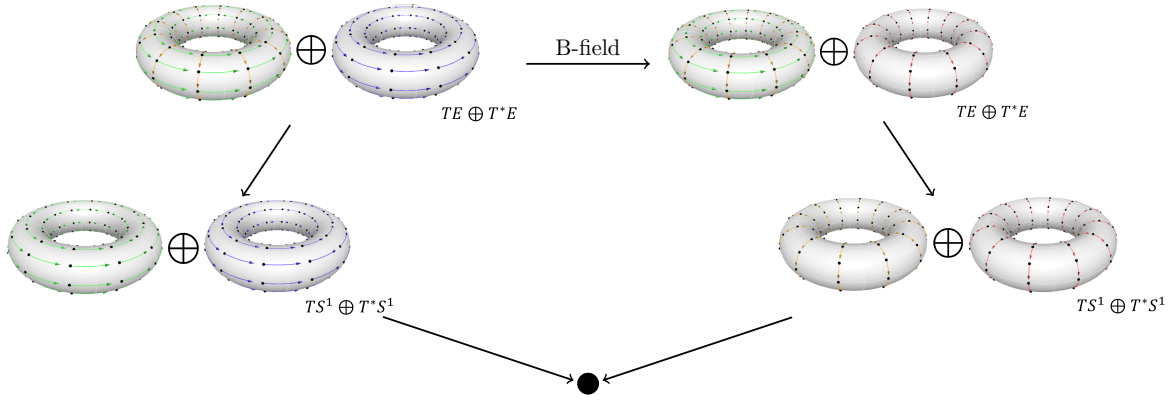


Figure 2: Schematic picture of T-duality. The base space is a point, the spaces are circles and the correspondence space is a Torus. After lifting the vector field on M one must do a B-field transformation to make the 1-form vertical so it can be pushed forward

The naive idea to define T -duality as a map of Courant algebroids is as follows: one would like to pullback sections of $(TM \oplus T^*M)/\mathbb{T}$ to sections of the Courant algebroid of the correspondence space and then pullback along the projection to $(T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ in order to obtain a map $(TM \oplus T^*M)/\mathbb{T} \rightarrow (T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$. However this naive process must be carefully defined, on one part the pullback of vector fields is not well defined and on the other hand the pushforward of forms only makes sense for basic forms. Thus to make sense of such procedure one requires to use the extra structure on the correspondence space. In an intuitive picture, depicted in (Figure 2): start with a vector field $X \in \Gamma(TM)/\mathbb{T}$ and a 1-form $\xi \in \Omega(T^*M)/\mathbb{T}$, one may lift the initial vector field to some vector field $\tilde{X} \in \Gamma(TM \times_B T\hat{M})$ which pushforward to X and pullback the form ξ to $\pi^*\xi$; then do an adequate B-field transformation, depending on \tilde{X} and K , in order to change $\pi^*\xi$ to a basic form. At last push-forward both structures to get a section of $(T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$. It turns out that there is only one \tilde{X} such that the above procedure makes sense, and thus the map is well defined.

The concrete map is the following

$$\begin{aligned} \varphi : (TM \oplus T^*M)/\mathbb{T} &\rightarrow (T\hat{M} \oplus T^*\hat{M})/\mathbb{T}, \\ \varphi(X + \xi) &= \hat{p}_*(\tilde{X} + p^*\xi - K(\tilde{X})), \end{aligned} \quad (4.9)$$

where $\tilde{X} \in T(M \times_B \hat{M})$ is the unique lift of $X \in \Gamma(TM)/\mathbb{T}$ satisfying $\iota_Y \xi - K(\tilde{X}, Y) = 0$, for all vector fields $Y \in \mathfrak{t}_M$, recall these are the tangent fields to the torus fiber on $M \rightarrow B$. Such a lift is well defined by non degeneracy of the form K . It is worth mentioning that the previous condition is in disguise the condition of $p^*\xi - K(\tilde{X})$ being basic, indeed it can be rewritten as $\iota_Y(p^*\xi - K(\tilde{X})) = 0$ and since both ξ and K are \mathbb{T} -invariant then $\mathcal{L}_Y(p^*\xi - K(\tilde{X})) = 0$, these are exactly the condition for a form to be basic.

Theorem 4.2.1 ([11]) For T -dual spaces M and \hat{M} the map $\varphi : (TM \oplus T^*M)/\mathbb{T} \rightarrow (T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ defined above is an isomorphism of Courant algebroids.

4.3 Trivial H-flux and Transport of Structures

As mentioned before in our case of interest in emergent gravity the H -flux is trivial, therefore we develop concretely the Courant isomorphism in this case. In the case of trivial H -flux every \mathbb{T} -bundle is T -dualizable. Indeed, recall that the obstruction to the existence of T -dual torus bundles is subject to a $G \in \Omega^2(B, \mathfrak{t})$ that satisfies (4.2), in the case of $H = 0$ it is easy to see that $G = 0$ satisfies such condition. Therefore the T -dual bundle exists and moreover by construction the Chern class of the bundle is $G = 0$, implying the bundle is trivial. The trivial bundle has a natural connection a_B given by $\hat{a} = \text{pr}_{\mathbb{T}}^*(a_{\mathbb{T}})$ of the unique invariant connection $a_{\mathbb{T}}$ on \mathbb{T}^3 via the projection

³This is usually known as the Maurer Cartan connection and in the case of torus bundles is given by $(d\theta^1, \dots, d\theta^n)$ where $d\theta$ is the unique normalized invariant form on \mathbb{T}^1 , i.e. $d\theta_g(X) = (R_{g^{-1}})_* X$

$\text{pr}_{\mathbb{T}} : B \times \mathbb{T} \rightarrow \mathbb{T}$. Since $a_{\mathbb{T}}$ is exact it follows \hat{a} is also closed and by definition the T -dual flux is given by $\hat{H} = da \wedge \hat{a}$. We conclude:

Proposition 4.3.1 Let M be a \mathbb{T} -bundle over B with trivial H flux, then the T -dual exists and is given by $B \times \mathbb{T}$ with $\hat{H} = da \wedge \hat{a}$.

This proposition exhibits the usual slogan that T -duality exchanges between trivial topology with non-trivial H -flux and non-trivial topology with trivial H -flux. To find the Courant algebroid consider $\tilde{X} + \tilde{\xi} \in \Gamma(TM \oplus T^*M)/\mathbb{T}$, a section and using the decomposition induced by the connection $a = (a^1, \dots, a^n)$, and the dual vector field $\partial = (\partial_1, \dots, \partial_n)$ decompose section as

$$\tilde{X} + \tilde{\xi} = X + f_i a^i + \xi + g^j \partial_j, \quad X \in \Gamma(TB), \quad \xi \in \Gamma(T^*B), \quad f_i, g^j \in C^\infty(B). \quad (4.10)$$

Now in this case $K = a \wedge \hat{a} = a_1 \wedge \hat{a}_1 + \dots + a_n \wedge \hat{a}_n$ and the unique lift \tilde{X} satisfying $\iota_{\partial_i} \xi - K(\tilde{X}, \partial_i) = 0 \forall i$ (i.e. has no a_i in its decomposition), is given by $\tilde{X} = X + f^i \partial_i + g^i \hat{\partial}_i$, where $g^i = g_i$, that is we rise the index just to make sense of the summation convention. Upon doing the B -field transformation

$$\begin{aligned} (\tilde{X} + p^* \tilde{\xi}) - K(\tilde{X}) &= (X + f^i \partial_i + g^i \hat{\partial}_i + \xi + g_i a^i) - (g_i a^i - f_i \hat{a}^i) \\ &= X + f^i \partial_i + g^i \hat{\partial}_i + \xi + f_i \hat{a}_i \end{aligned} \quad (4.11)$$

At last taking the pushforward along \hat{p} , i.e. ‘integrating’ the vector field in the ∂_i variables we obtain $\varphi : (TM \oplus T^*M)/\mathbb{T} \rightarrow (T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ is given by

$$X + f_i a^i + \xi + g^j \partial_j \quad \mapsto \quad X + g_i a^i + \xi + f^j \partial_j. \quad (4.12)$$

That is T -duality in this case is given by an interchange of the variables $f_i \leftrightarrow g^i$ which can be considered as an interchange of position and momenta coordinates.

Given the Courant algebroid isomorphism one may transport Dirac structures or generalized structures from the Courant algebroid $(TM \oplus T^*M)/\mathbb{T}$ to $(T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ in the following way: A Dirac (generalized complex) structure on M is determined by $C \subset (TM \oplus T^*M)/\mathbb{T}$ a maximally isotropic subvariety of $(TM \oplus T^*M)/\mathbb{T}$ (the respective complexification), upon doing the Courant algebroid isomorphism the image maps again to a maximally isotropic subvariety of $(T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ with respect to the H -twisted bracket (the complexification) thus defining a twisted H -Dirac (generalized complex) structure for \hat{M} .

To show how this procedure works in practice, and which will be of importance later, consider the generalized complex structure defined by a symplectic form $\omega \in M$ on a \mathbb{T}^2 -torus bundle M . Given the decomposition $T^*M = T^*B \oplus \langle a_1 \rangle \oplus \langle a_2 \rangle$ the symplectic structure decomposes as

$$\omega = \omega_0 a^1 \wedge a^2 + \omega_{1,1} \wedge a^1 + \omega_{1,2} \wedge a^2 + \omega_2, \quad (4.13)$$

where $\omega_0 \in C^\infty(B)$, $\omega_{1,1}, \omega_{1,2} \in \Omega^1(B)$ and $\omega_2 \in \Omega^2(B)$. The generalized complex structure of ω is determined by its graph $C_\omega = \{\tilde{X} - i \iota_{\tilde{X}} \omega \mid X \in \Gamma(M)\}$, in which a general element is given explicitly in the decomposition on vertical and horizontal parts by

$$\begin{aligned} &X + f_1 \partial_1 \\ &+ f_2 \partial_2 \\ &+ \omega_2 - f_1 \omega_{1,1} - f_2 \omega_{1,2} \\ &+ (-\omega_0 f_2 + \omega_{1,1}(X)) a_1 \\ &+ (\omega_0 f_1 + \omega_{1,2}(X)) a_2 \end{aligned} \quad (4.14)$$

Under the Courant algebroid isomorphism a general element of C_ω , maps to an element

$$\begin{aligned} &X + [i\omega_0 f_2 - i\omega_{1,1}(X)] \tilde{\partial}_1 \\ &+ [-i\omega_0 f_1 - i\omega_{1,2}(X)] \tilde{\partial}_2 \\ &+ i[\iota_X(\omega_2) - \omega_{1,1} f_1 - \omega_{1,2} f_2] \\ &+ f_1 \tilde{a}_1 \\ &+ f_2 \tilde{a}_2. \end{aligned} \quad (4.15)$$

We wish to find a generalized complex structure of type 0, i.e. a symplectic form with a B-field (2-form) $\tilde{\omega}$, \tilde{B} , such that the graph corresponding to $e^{B+i\omega}$ is given by the elements defined by $\varphi(C_\omega)$, i.e elements of the form of equation (4.15). Again we consider a decomposition of $\tilde{\omega}$ and \tilde{B} with respect to the decomposition $T^*M = T^*B \oplus \langle a_1 \rangle + \langle a_2 \rangle$. With such a decomposition a general element of the graph, of the unknown, $e^{\tilde{B}+i\tilde{\omega}}$ is:

$$\begin{aligned}
& X + \tilde{f}_1 \tilde{\partial}_1 \\
& + \tilde{f}_2 \tilde{\partial}_2 \\
& - i \left[\iota_X \tilde{\omega}_2 - \tilde{\omega}_{1,1} \tilde{f}_1 - \tilde{\omega}_{1,2} \tilde{f}_2 + \iota_X (\tilde{B}_2) - \tilde{B}_{1,1} \tilde{f}_1 - \tilde{B}_{1,2} \tilde{f}_2 \right] \\
& + \left[i \tilde{\omega}_0 \tilde{f}_2 - i \tilde{\omega}_{1,1}(X) - \tilde{B}_0 \tilde{f}_2 + \tilde{B}_{1,1}(X) \right] \tilde{a}_1 \\
& + \left[-i \tilde{\omega}_0 \tilde{f}_1 - i \tilde{\omega}_{1,2}(X) + \tilde{B}_0 \tilde{f}_1 + \tilde{B}_{1,2}(X) \right] \tilde{a}_2.
\end{aligned} \tag{4.16}$$

Taking $\tilde{f}_1 = i\omega_0 f_2 - i\omega_{1,1}(X)$ and $\tilde{f}_2 = -i\omega_0 f_1 - i\omega_{1,2}(X)$, the ∂_1 and ∂_2 components agree (notice $X + \tilde{f}_1 \partial_1 + \tilde{f}_2 \partial_2$ is still a general element of TM). The condition \tilde{C}_ω to be the graph of $e^{\tilde{B}+i\tilde{\omega}}$ implies an equality of the expressions of (4.15) and (4.16), which in turn sets a system of equations that can be used to find the explicit description of $\tilde{\omega}$ and \tilde{B} .

Equation arising from the \tilde{a}_1 components and coefficients of f_1 is:

$$1 = \tilde{\omega}_0 \omega_0 + i \tilde{B}_0 \omega_0 \quad \Rightarrow \quad \tilde{\omega}_0 = \frac{1}{\omega_0} \quad \tilde{B}_0 = 0, \tag{4.17}$$

where we used that $\tilde{\omega}$ is a symplectic form implying $\tilde{\omega}_0$ should be real. Analogously, the equations arising from the contraction with X in the a_1 and a_2 components are:

$$0 = \tilde{\omega}_0 \omega_{1,2}(X) - i \tilde{\omega}_{1,1}(X) + \tilde{B}_{1,1}(X) \quad \Rightarrow \quad \tilde{\omega}_{1,1} = 0, \quad \tilde{B}_{1,1} = \frac{-\omega_{1,2}}{\omega_0}. \tag{4.18}$$

$$0 = -\tilde{\omega}_0 \omega_{1,1}(X) - i \tilde{\omega}_{1,2}(X) + \tilde{B}_{1,2}(X) \quad \Rightarrow \quad \tilde{\omega}_{1,2} = 0, \quad \tilde{B}_{1,2} = \frac{\omega_{1,1}}{\omega_0}. \tag{4.19}$$

At last using the result from (4.18) and (4.19) to the equation obtained from the contraction with X on the basic 2-form we get:

$$\tilde{\omega}_2(X) = \omega_2(X) - \frac{\omega_{1,1}(X)\omega_{1,2}}{\omega_0} + \frac{\omega_{1,2}(X)\omega_{1,1}}{\omega_0} - i \tilde{B}_2(X), \tag{4.20}$$

which again using that $\tilde{\omega}_2$ should be real gives:

$$\tilde{\omega}_2 = \omega_2 - \frac{\omega_{1,1} \wedge \omega_{1,2}}{\omega_0}, \quad \tilde{B}_2 = 0. \tag{4.21}$$

We conclude that the T -dual of a symplectic form ω on M with symplectic fibers ($\omega_0 \neq 0$) is again a (twisted) symplectic form and is given by $\tilde{\omega} = \tilde{\omega}_0 + \tilde{\omega}_{1,1} \wedge a^1 + \tilde{\omega}_{1,2} \wedge a^2 + \tilde{\omega}_2$ where each component is given by the equations (4.17) - (4.21).

5 T-Duality and Emergent Gravity

5.1 Noncommutative U(1)-theory, emergent gravity and generalized geometry

Emergent gravity as approached by [2, 3, 26] is the consequence of the Darboux theorem in symplectic geometry, which should be thought as an analogue of the equivalence principle in Riemannian geometry in the symplectic disguise. Intuitively in the presence of an electromagnetic force F the equivalence principle, via the Darboux theorem, induces a diffeomorphism which locally leads to an equivalent theory in which one can eliminate the force F but introduces dynamical variables to the metric, thus the metric ‘emerges’ from the data of a symplectic form ω and an electromagnetic force F . Thus in our framework of the T -dual of emergent gravity we have a T -dualizable Dp-brane with constant background metric g and a constant background B-field, i.e. the following data: M is a \mathbb{T} -bundle with a connection $a = (a_1, \dots, a_n)$ and trivial H -flux, moreover M is assumed to be a symplectic manifold with \mathbb{T} -invariant symplectic form B given by the background B-field, together with a line bundle with

connection $(L \rightarrow M, A)$ such that $dA = F$ (see the diagram appearing in glossary in the introduction).

We start with a short review detailing the relationship between non-commutative $U(1)$ -theory and the emergent gravity. A (semi-classical) non-commutative gauge theory is described by the deformation of a $U(1)$ -theory by a Poisson structure: the data of such a theory is given by a non-commutative line bundle $\tilde{L} \rightarrow M$ together with a non-commutative connection A^{NC} with a non-commutative field strength F^{NC} , it has been already shown, see for example [27, 28], that the data of a non-commutative $U(1)$ -theory can be equivalently described by a commutative gauge theory with a Poisson bi-vector θ , i.e. a line bundle $L \rightarrow M$ with a connection form A with curvature $dA = F$ and the Poisson bi-vector θ , the relationship between commutative and non-commutative field strengths is given by the *Seiberg-Witten map*.

$$F^{NC} = (1 + \theta F)^{-1} F. \quad (5.1)$$

The non-commutative action [1] is a Moyal deformation of the Dp-Brane with flat metric g on a noncommutative target space with electromagnetic field F^{NC}

$$S^{NC} = \frac{1}{g_s^2} \int d^n x \sqrt{g} g^{ik} g^{jl} \text{Tr}((F^{NC})_{ij} * (F^{NC})_{kl}) \quad (5.2)$$

Upon using the dictionary of commutative and non-commutative structures dictated by the *Seiberg-Witten map*[1], and considering an expansion in the Poisson structure as $\theta = 1 + \hbar\theta^{(1)} + \mathcal{O}(\hbar^2)$, the semi-classical non-commutative action (5.2) is up to first order equivalent to an action of a commutative Dp-brane with field strength F and emergent metric G^{em} .

$$S = \frac{1}{G_s^2} \int d^n x \sqrt{G^{\text{em}}} (G^{\text{em}})^{ik} (G^{\text{em}})^{jl} \text{Tr}(F_{ij} F_{kl}) + \mathcal{O}(\hbar^2) \quad (5.3)$$

The emergent metric G^{em} up to first order in the deformation parameter θ is given by

$$G^{\text{em}} = g(1 + F\theta^{(1)}). \quad (5.4)$$

Now from the generalized geometry picture the information of a commutative Dp-brane with background metric g and electromagnetic field F can be encoded in a generalized metric $g + F$, meanwhile the deformation parameter can be encoded by a θ -transformation. Now with $\theta = B^{-1}$ being the deformation parameter we can show that G can be obtained uniquely from the data θ , F and g . Indeed if we start with a generalized metric $g + F$ upon doing a B -field transform e^{-F} we obtain another generalized metric

$$g \xrightarrow{e^F} g + F, \quad (5.5)$$

which after performing a θ -transform with $e^{-\theta}$ leads to a generalized metric $G + \Phi$. This is in agreement with the open-closed string duality. In this article we start from a closed string background (g, B) , and pick a poisson bivector θ to determine the open string variables (G, Φ) .

$$g + F \xrightarrow{e^{-\theta}} G + \Phi, \quad G = g(1 - F\theta)^{-1}, \quad \Phi = 0. \quad (5.6)$$

Indeed, upon doing a θ -transformation by $(B + F)^{-1}$ to the generalized metric described by g we obtain a new generalized metric G^θ described by the metric G satisfying the equality

$$G = g - Fg^{-1}F, \quad G\theta = -Fg^{-1}, \quad (5.7)$$

where we can find G explicitly as

$$G = g + G\theta F \quad \Rightarrow \quad G = g(1 - \theta F)^{-1}, \quad (5.8)$$

which up to first order expansion in θ is $g(1 + \theta^{(1)}F)$ which is exactly the emergent metric (5.4). Thus emergent gravity from the generalized geometry framework arises from performing a θ -transformation e^θ after a B -field transformation e^F . This also explains the Seiberg-Witten formula for the non-commutative field strength F^{NC} since after applying with e^θ the Dirac structure defined by the 2-form F is again a Dirac structure defined by a 2-form if and only if $1 + \theta F$ is invertible, moreover in such case is given by the 2-form $F^{NC} = (1 + \theta F)^{-1}F$. We now turn to a simple yet important example that of flat compact spacetimes, these were the first instances where T -duality appeared from the generalized geometric framework and are well behaved mathematical operation.

5.2 Flat Spacetimes

The theory of emergent gravity was initially studied on trivial spacetimes in [3, 27], and were further explored in other geometries as well [29] where they were studied together with their topological relationship with the spaces. For the purpose of the present paper we shall study emergent gravity theories in toroidal compactification in the light of its link to T -duality. Thus in this sub-section we study toroidally compactified Dp-Branes on a space with (flat) background metric g and constant B -field.

From the T -duality point of view developed in the previous sections the framework is obtained by setting the base space B as a point: in this case the total space and its dual are diffeomorphic to a torus, $M \cong \mathbb{T}$ and $\tilde{M} \cong \mathbb{T}$; the space of invariant forms and multi-vector fields are given exactly by constant forms and multi-vector fields on the space M or \tilde{M} ; there is no appearance of flux H on the T -dual by proposition 4.3.1 and the fact that $da = 0$, since any form on the point manifold vanishes. Moreover we assume the B -field is closed and invertible which implies M is a symplectic manifold with symplectic form B , thus an even dimensional Tori. We can describe concretely the transport of generalized metrics on M via the Courant algebroid isomorphism, indeed let \mathcal{G} be a generalized metric on M described by the pair (g, B) (as in section 3) then the isotropic subvariety defining the generalized metric is given by the eigenbundle $E_- = \{X - g(X) + B(X) \mid X \in TM\}$, now the decomposition of $TM \oplus T^*M$ defined by the connection, on the eigenbundle is given by elements of the form

$$X^k \partial_k + [(g + b)_{ij} a^i \wedge a^j](X^k \partial_k) = X^i \partial_i + (g + b)_{ik} X^k a^i, \quad (5.9)$$

which under the Courant isomorphism are mapped to

$$X_i a^i + (g + b)_k^i X^k \partial_i = \tilde{X}^i \partial_i + (g + b)_i^{k-1} \tilde{X}_k a^i, \quad (5.10)$$

where $\tilde{X}^i = (g + b)_k^i X^k$ and we have made use of dummy index for raising and lowering due to the emergent T -duality exchange between $a^i \leftrightarrow \partial_i$. We conclude that under the Courant isomorphism the generalized metric is transformed as $\varphi : \mathcal{G} \mapsto \mathcal{G}^{-1}$, where we consider this inversion as the block matrix inversion plus inversion of g and B . This T -duality action on generalized metrics has already been considered in the literature, see for instance [30] for an extensive survey. The concrete relationship between the metric $g + (B + F)$ and the T -dual counterpart $\hat{g} + \widehat{B + F}$ is given by comparing the matrix components of the corresponding generalized metrics

$$\hat{g} = g - (B + F)g^{-1}(B + F), \quad \widehat{(B + F)}^{-1} = -(B + F)g^{-1}. \quad (5.11)$$

We have shown why it is natural to consider the appearance of Poisson structures as the counterpart of B -field transformation upon T -dualizing.

Consider the decomposition of $\mathcal{G} = g + B + F$ as $e^{B+F}g$, now T -duality acts on the map e^{B+F} by conjugation, $\varphi e^{B+F} \varphi^{-1}$ which is a θ -transformation given by $e^{(B+F)^{-1}}$. The θ -transformation by $(B + F)^{-1}$ acts in the generalized metric g already described and we obtain a new generalized metric \mathcal{G}^θ described by the metric G , a 2-form $\Phi = 0$ and Poisson vector θ satisfying the equalities

$$G = g - (B + F)g^{-1}(B + F), \quad G\theta = -(B + F)g^{-1}. \quad (5.12)$$

Realizing that the T -dual of the generalized metric g is g itself and then comparing with equation (5.11) we obtain the equality

$$e^\theta \varphi(g) = \varphi(e^{B+F}g), \quad (5.13)$$

Thus T -duality acts on B -field transform by remolding them into θ -transformation. Therefore under the generalized geometric framework T -duality gives a natural approach to the already well known duality between open string with non-zero B -field and closed strings with non-commutative gauge theories with deformation parameter $\theta = (B + F)^{-1}$ [27, 31, 32].

We have studied how non-commutative structures on strings/branes may be understood in the light of T -duality from the generalized geometry framework, we now wish to study the T -dual avatar of an emergent metric. Recalling the connection of generalized geometry to emergent gravity narrated in the previous section we know that emergent gravity can be described by the composition of θ -transformation and B -field transformation i.e. $e^\theta \cdot e^F$, where

$\theta = B^{-1}$ is the inverse of a background symplectic B -field. Upon T -dualizing we have that in the T -dual picture, emergent gravity is given by the following commutative diagram at the level of generalized metrics

$$\begin{array}{ccc} g & \xrightarrow{\varphi} & g \\ \downarrow e^F & & \downarrow e^{F^{-1}} \\ g + F & \xrightarrow{\varphi} & (g + F)^{-1} \\ \downarrow e^\theta & & \downarrow e^{\theta^{-1}} \\ G & \xrightarrow{\varphi} & \hat{G} \end{array},$$

Notice that the right side of the diagram corresponding to the T -dual does not describe emergent gravity since it is given by a B -field transform after a θ -transform, while emergent gravity is described by such transformation in the inverse order. To fix this we use the fact that the composition $e^\theta e^F$ can be rewritten as

$$e^\theta e^F = e^{F'} O_N e^{\theta'}, \quad (5.14)$$

for F' , θ' and N defined as

$$\theta' = (1 + \theta F)^{-1} \theta = \theta (1 + F \theta)^{-1} \quad (5.15)$$

$$F' = F (1 + \theta F)^{-1} = (1 + F \theta)^{-1} F \quad (5.16)$$

$$N = 1 + F \theta. \quad (5.17)$$

The map N of Courant algebroids is a map induced by a diffeomorphism φ_N of the manifold M (which in this case is diffeomorphic to \hat{M}) which on the Poisson structures induces an isomorphism $(M, \theta) \cong (M', \theta')$. This diffeomorphism is obtained via the Moser trick and can be considered as a Darboux coordinate transformation which is crux of emergent gravity, it is in fact the diffeomorphism mapping between the symplectic structures $B = \theta^{-1}$ and $B' = \theta'^{-1}$ which is at the heart of our construction. For a detailed exposition of this result we refer the readers to [6].

After T -dualizing (5.14) we obtain the equality

$$\begin{aligned} \varphi e^\theta e^F \varphi^{-1} &= \varphi e^{F'} O_N e^{\theta'} \varphi^{-1} \\ &= \varphi e^{F'} \varphi^{-1} \varphi O_N \varphi^{-1} \varphi e^{\theta'} \varphi^{-1} \\ &= e^{(F')^{-1}} O_N e^{(\theta')^{-1}} \end{aligned} \quad (5.18)$$

Therefore, up to a change of coordinates given by the diffeomorphism φ_N , we conclude that the T -dual picture to emergent gravity is given by the background electromagnetic field \hat{F} and a non-commutative structure with deformation parameter $\hat{\theta}$ given by

$$\begin{aligned} \hat{F} &:= (\theta')^{-1} \\ \hat{\theta} &:= (F')^{-1} \end{aligned} \quad (5.19)$$

Moreover if \hat{F} is pre-quantizable then there exists a line bundle $\hat{L} \rightarrow \hat{M}$ with connection \hat{A} such that $\hat{F} = d\hat{A}$, thus in this case there exists a $U(1)$ -theory such that the 2-form \hat{F} can be considered as the electromagnetic force of a ‘ T -dual electromagnetic theory’. Notice that the T -dual electromagnetic force is associated to the original B -field B , while the T -dual deformation parameter is related to the original electromagnetic force F . We conclude that in a flat spacetime with a pre-quantizable \hat{F} the T -dual picture of emergent gravity is again an emergent gravity theory. However, as will be further explored in the next section, this may not hold true on non-flat spacetime where, in our framework, we have a non-trivial base manifold.

5.3 Non-flat Spacetimes

In our framework on non flat spacetimes we may have a non trivial base manifold B . Thus, as explained before, the T -dual manifold has in general a non trivial H -flux \hat{H} which depends on the curvature of the original torus bundle connection. In the presence of a background H -flux the Dorfmann bracket is twisted and maximally isotropic subvarieties do no longer represent Dirac structures or generalized complex structures but their H -twisted

counterparts, in particular in our context the T -dual generalized metric is twisted by the 3 form \hat{H} . In contrast with the case of flat spacetimes, due to the existence of a twist, emergent gravity framework is no longer available in the T -dual framework, therefore one is not able to express the T -dual theory of emergent gravity as another emergent gravity theory. This obstacle may be avoided in two ways: one may either consider just \mathbb{T} -bundles with flat connection (thus there is no appearance of H -flux on the T -dual); or begin with considering both the initial and final T -dual bundles on the same footing with a background H -flux. However the first approach restricts greatly the possible interesting geometries to consider, and even with this restriction the *Courant algebroid isomorphism* when $B \neq 0$ is highly nontrivial (in comparison to the flat spacetime scenario) which makes calculation in the emergent gravity setting intractable. Meanwhile, the study of both T -duality and emergent gravity with a background H -flux is a field still in progress and a vast number of literature has already been dedicated towards understanding the phenomena both in physics and mathematics.

A general formula was derived for the topology and H-flux of the T-dual for a type II compactification in [14]. Summarizing, topological T-duality uses the following topological data: A principal $U(1)$ bundle $\pi : M \rightarrow B$, together with a pair of cohomology classes (F, H) The class $F \in H^2(B, \mathbb{Z})$ is the first Chern class, and determines the isomorphism class of the bundle, whereas the class $H \in H^3(M, \mathbb{Z})$ is the cohomology class of the curvature of the B -field. T-duality intermixes F and H . For an account of this topological version of T duality between pairs of principal $U(1)$ -bundle (also extended for principal \mathbb{T}^n -bundles) equipped with degree-3 integral cohomology class from a twisted K-theory point of view one may look at the works of Bunke et. al [33].

Another mathematical point of view on the action of T-duality on generalized complex structures on vector spaces and torus bundles with trivial H-flux mimicking our context appears in the work of Ben-Bassat [34]. Later on Gualtieri and Cavalcanti [35] have treated the most general case of T-duality with non-trivial H-flux for higher rank affine torus bundles such that $\iota_X \iota_Y H = 0 \forall X, Y$ tangent to the fibers. It is worth mentioning here that in Mathai and Rosenberg's earlier works [36] this restriction on H flux was not exploited yet they made sense of the T dual manifold by interpreting it as a noncommutative space. More precisely, their work is based on T-dualizing along a \mathbb{T}^2 with non-zero H-flux that yields a fibration by noncommutative tori. In [37] noncommutative torus fibrations were shown to be the open string version of T-folds and non-geometric T-dual of \mathbb{T}^3 with uniform H-flux was embedded into a generalized complex \mathbb{T}^6 .

Interested readers may find some relevant discussion concerning nontrivial H-flux and related noncommutative gerbe in [38]. T duality in the heterotic string setting is different from the conventional T-duality in the sense that it introduces a gauge bundle with connection leading to more complications namely the modified topological conditions on the existence of T-dual. More importantly, in this case the H-flux is no longer closed and does not correspond to a gerbe globally. All these investigations may be found in [39].

Geometry of double field theory [40] has a structural similitude to that of emergent gravity on Calabi Yau manifolds. Recently it was found in [12] by Yang how emergent gravity leads to a beautiful picture of mirror symmetry and the variety of six-dimensional manifolds emergent from noncommutative $U(1)$ gauge fields is doubled as an artifact of Hodge theory for the deformation of symplectic and dual symplectic structures in six spacetime dimensions. But all these diversified approaches go beyond the scope of the present work and will be studied in future works [41].

Nevertheless, one can still calculate the explicit description of the generalized metric following the procedure described in section 4.3. although there might not be an emergent gravity description for the T -dual theory. To demonstrate this example in a concrete case we will consider a spacetime M which is a \mathbb{T}^2 -fibration over an arbitrary base manifold B . To find the T -dual generalized metric we first decompose, according to the direct sum $T^*M \cong T^*B \oplus \langle a_1 \rangle \oplus \langle a_2 \rangle$ and $TM \cong TB \oplus \langle \partial_1 \rangle \oplus \langle \partial_2 \rangle$ given by the connection $a = (a_1, a_2)$, the background flat metric g , the electromagnetic field 2-form F and the Poisson structure θ describing the emergent gravity. In the \mathbb{T}^2 -case these decomposition are given by

$$\begin{aligned} g &= a_1 \otimes a_1 + a_2 \otimes a_2 + g_2 \\ F &= F_{0,12} a_1 \wedge a_2 + F_{1,1} \wedge a_1 + F_{1,2} \wedge a_2 + F_2 \\ \theta &= \theta_{0,12} \partial_1 \wedge \partial_2 + \theta_{1,1} \wedge \partial_1 + \theta_{1,2} \wedge \partial_2 + \theta_2. \end{aligned} \tag{5.20}$$

In this decomposition of the elements we have used 2 under-scripts for notational purpose where the first under-script denotes the degree of the form on B and the second the connection which it is paired to, for example $F_{1,2}$ is a basic 1-form and may be considered as $F_{1,2} \in \Omega^1(B)$ and in the decomposition of F appears wedged to a_2 . Notice that F_2 and g_2 are 2-forms in the base which defines a B -field and a Riemannian metric on B respectively.

Notice also that the special decomposition of g have been used due to the fact that it is flat. Given the previous decomposition and recalling $\partial_i(a^j) = \delta_i^j$ one finds decomposition of $F\theta$ (considered as a (1,1)-tensor) to be

$$\begin{aligned}
F\theta = & (F_{0,12}\theta_{0,12} + \theta_{1,1}(F_{1,1}))a_1 \otimes \partial_1 + (F_{0,12}\theta_{0,12} + F_{1,2}(\theta_{1,2}))a_2 \otimes \partial_2 \\
& + F_{1,1}(\theta_{1,2})a_1 \otimes \partial_2 + F_{1,2}(\theta_{1,1})a_2 \otimes \partial_1 \\
& + a_1 \otimes [F_{0,12}\theta_{1,2} + \theta_2(F_{1,2})] + a_2 \otimes [-F_{0,12}\theta_{1,1} + \theta_2(F_{1,2})] \\
& + [\theta_{0,12}F_{1,2} + F_2(\theta_{1,1})] \otimes \partial_1 + [-\theta_{0,12}F_{1,1} + F_2(\theta_{1,1})] \otimes \partial_2 \\
& + F_{1,1} \otimes \theta_{1,1} + F_{1,2} \otimes \theta_{1,2} + F_2\theta_2,
\end{aligned} \tag{5.21}$$

where for example $\theta_2(F_{1,2})$ means the contraction of the bi-vector field θ_2 with the basic 1-form part of F . Using the decomposition in (5.21) the emergent metric G (given by equation 5.4) is described by the decomposition

$$G = G_{0,11}a_1 \otimes a_1 + G_{0,22}a_2 \otimes a_2 + G_{0,12}a_1 \otimes a_2 + G_{1,1} \otimes a_1 + G_{1,2} \otimes a_2 + G_2, \tag{5.22}$$

with the following components

$$\begin{aligned}
G_{0,11} &= 1 + F_{0,12}\theta_{0,12} + \theta_{1,1}(F_{1,1}), \\
G_{0,22} &= 1 + F_{0,12}\theta_{0,12} + \theta_{1,2}(F_{1,2}), \\
G_{0,12} &= F_{1,1}(\theta_{1,1}) + F_{1,2}(\theta_{1,2}), \\
G_{1,1} &= F_2(\theta_{1,1}) + \theta_{0,12}F_{1,2} + g_2(\theta_2(F_{1,1}) + F_{0,12}\theta_{1,2}) \\
G_{1,2} &= F_2(\theta_{1,2}) - \theta_{0,12}F_{1,1} + g_2(\theta_2(F_{1,2}) - F_{0,12}\theta_{1,1}) \\
G_2 &= 1 + F_{1,1} \otimes g_2\theta_{1,1} + F_{1,2} \otimes g_2\theta_{1,2} + g_2F_2\theta_2
\end{aligned} \tag{5.23}$$

Now one may T -dualize such emergent metric using the Courant algebroid isomorphism described by T-duality, the details of how to obtain the T -dual of an arbitrary generalized metric on a \mathbb{T}^2 -fibration are given in the appendix, and for convenience we just state the final result for the emergent gravity G (with zero B -field component) case:

$$\begin{aligned}
\hat{G}_{0,11} &= \frac{G_{0,22}}{G_{0,11}G_{0,22} - G_{0,12}^2}, \\
\hat{G}_{0,22} &= \frac{G_{0,11}}{G_{0,11}G_{0,22} - G_{0,12}^2}, \\
\hat{G}_{0,12} &= \frac{G_{0,12}}{G_{0,11}G_{0,22} - G_{0,12}^2}, \\
\hat{G}_{1,1} &= 2 \frac{G_{0,22}G_{1,1} - G_{0,12}G_{1,2}}{G_{0,11}G_{0,22} - G_{0,12}^2}, \\
\hat{G}_{1,2} &= 2 \frac{G_{0,11}G_{1,2} - G_{0,12}G_{1,1}}{G_{0,11}G_{0,22} - G_{0,12}^2}, \\
\hat{G}_2 &= G_2 + \frac{G_{0,12}(G_{1,1} \otimes G_{1,2}) - G_{0,22}(G_{1,1} \otimes G_{1,1}) - G_{0,11}(G_{1,2} \otimes G_{1,2})}{G_{0,11}G_{0,22} - G_{0,12}^2}.
\end{aligned} \tag{5.24}$$

In the case of a non-flat spacetime the above set of equations may be considered as the defining basis for the T -dual emergent gravity on a \mathbb{T}^2 -fibration, however the authors are still trying to find a physical framework in which the above formulae can naturally be understood [41].

6 Conclusion

Let us begin by highlighting the principal findings in the present article. After setting up a generalized geometric viewpoint of emergent gravity from the scratch and outlining the reduction process of exact Courant algebroid in the emergent gravity theory, we have proceeded to look for a T dual avatar of the theory. In the hunt for a T dual theory we have used the Gualtieri-Cavalcanti (GC) construction of isomorphism between Courant algebroids and have shown how to transport geometric structures between the torus bundles in action.

While most of the contents of sections 2 and 3 (except for subsection 2.2 and 2.3 that deals with atiyah algebroid in emergent gravity and the reduction of exact courant algebroid therein) are for pedagogical backgrounds and can

be thought of as a review on the subject, later sections of the manuscript are dedicated towards explicating our main results. In section 4 we begin by recapitulating the Bouwknegt-Evslin-Hannabuss-Mathai [14] construction of topological T duality followed by a short summary of what is known as isomorphism of courant algebroids which is the crux of topological T duality. In section 4.3 we have concretely demonstrated the mechanism of T duality à la GC map [11, 35] by transporting a symplectic form on a torus bundle with symplectic fiber to its T dual counterpart. After identifying the bilinear form on sections $(\mathcal{X}, \Xi) \in (TM \oplus T^*M)/\mathbb{T}$ with a quadruple (X, f, ξ, g) as $\mathcal{X} = X + f_i a^i$ and $\Xi = \xi + g^j \partial_j$ and defining a map $\varphi : (TM \oplus T^*M)/\mathbb{T} \rightarrow (T\hat{M} \oplus T^*\hat{M})/\mathbb{T}$ as in (4.12) we re-express T-duality in emergent gravity as an interchange of variables $f_i \leftrightarrow g^i$ which in turn induces a frame duality $\partial_i \rightarrow a^i$ in further calculations and this can be considered as an interchange of position and momenta coordinates. In the end we find that under the GC map a symplectic form goes to another symplectic form related to the initial one via the set of equations (4.17) - (4.21) that we have written down explicitly for the emergent gravity with trivial H-flux.

Section 5, which is probably the most crucial part of the present endeavour, highlights the connection between emergent gravity and T duality. Taking the inspiration from the celebrated work of Seiberg and Witten [1] establishing a duality between semi-classical non-commutative degrees of freedom in a gauge theory to its commutative counterpart, we show how for an emergent gravity theory in toroidal compactification, T duality acts on background B-fields changing them into θ - transformation. Without the B-fields Gualtieri-Cavalcanti map does not change the Riemannian metric, but in presence of it emergent gravity can be described by a sequence of operations namely the B-field transformation first and then the θ transformation on the metric, while on the T-dual side, the sequence gets interchanged and this feature is well captured in the commutative diagram that appears in section 5.2 of our article. We also show why under a GC map a diffeomorphism goes to a diffeomorphism thus indicating the T-dual avatar could be another gravity theory, as well. All of this works because underneath there is the mechanism of *Courant isomorphism* which is nothing but the T duality acting between torus bundles under investigation (also Courant algebroids) changing the generalized metric to its inverse under the GC isomorphism map. If this concept is absorbed really well then rest of the equations of the subsection 5.2 starting with (5.14) till (5.19) make sense without much effort.

The final subsection 5.3 is an attempt to generalize our formalism for any arbitrary base manifold B . This can be achieved easily following the same routine as of the previous section. Starting with the initial data (5.20) of the emergent metric for a \mathbb{T}^2 fibration on a generic base manifold B , we were able to obtain the precise mathematical expressions for the components of the emergent metric (see eqn.5.23) and its T-dual counterpart (see eqn. 5.24). This in our opinion form the novel aspects of the current work.

The scenario with H-flux is bit tricky, because in this case, we have objects like H-twisted symplectic or H-twisted poisson. In the case of twisted symplectic structure this is not a 2-form that is closed but d acting on the 2-form is equal to the flux ($H = dB$), and since it is not symplectic it does not lead to a poisson structure. On the other hand, the poisson structure does not satisfy the Jacobi identity (actually it does but upto some dH). With H flux there is no symplectic structure and so under T duality we cannot change it to a poisson. Although the problem is not with the generalized geometry picture. We can consider H twisted objects using GC map [35]. What happens is that after a twist, isotropic integrable subvarieties under a Dorfman bracket (2.9) are no more integrable and hence the graphs (3.9) do no longer represent the generalized metric. By considering T-duality for strings (or Dp branes) moving in a geometric background i.e. in presence of curvature and H fluxes, it is possible to arise at a situation where the string (or Dp branes) is coupled to what is known in the literature as, non-geometric fluxes [42]. It would be interesting to unravel the full geometry behind these non-geometric fluxes in the context of emergent gravity.

For emergent gravity we have a framework [2] where we start with a noncommutative $U(1)$ theory that is equivalent to a commutative $U(1)$ theory where the metric changes. The commutative gauge theory is given by a line bundle, its curvature and a poisson structure, these are the initial data. In the case of H flux we don't have any of these data, we only know them upto a twist in H . If we want to know what should be the avatar of emergent gravity in the twisted case we have to upgrade all of the data to incorporate twist. What is gauge theory on a space with H-flux? What is a poisson structure on a space with H -flux? And how are they related to give rise to a gravity theory. These are the immediate questions that need to be addressed and we don't know of any of these structures in emergent gravity. What is a noncommutative line bundle in a space with H flux? In the case $H = 0$ we have a trivial gerbe that is equivalent a $U(1)$ line bundle, but when we have nontrivial flux this line bundle has to be replaced by a $U(1)$ gerbe and in this case the H-twisted poisson structure will give us a noncommutative (NC) $U(1)$ gerbe. One may try to write down the action of this theory and further probe the field theoretic nature of such theory given by $U(1)$ gerbes. These theories are known as categorical field theories (and by nature higher gauge theories). The noncommutative field theories that are locally described by the NC gerbes may give us an analogue of SW

map. In the works of Hull et. al [43] we find how to lift the connection between holomorphic line bundle and kahler geometry to bi-holomorphic line bundle and generalized complex geometry in the context of T-duality. In order to concretely formulate emergent gravity in the gerby geometric language readers can also look in [44] where the authors have presented nonabelian gerbes as a higher geometric generalization of principal line bundles that appear in our theory. It is worth mentioning that there is a very recent work in the literature [45] that deals with topological T duality in this gerby framework and we believe this might turn out to be quite useful for our future [41] study. But all of these are quite nontrivial and beyond the scope of present project, but we shall come back to it soon. This is doable if one works with the local formula of what a gerbe is, especially in the context of emergent gravity.

For a particular case with a nontrivial $H = dB$, we can still construct emergent gravity. For example, if $H = B \wedge b$ where b is the so-called Lee form [46], it is known as a locally conformal symplectic (LCS) manifold [47]. See, for example, appendix A of [48] for an important feature of this manifold that the local structure is exactly the same as the symplectic manifold and so we can have local Darboux charts. We think LCS manifold is necessary to describe the inflationary universe (and black holes) in the context of emergent gravity. In this case the B-fields generate a more general diffeomorphism symmetry than the symplectic B-fields. There is every reason to believe that the most general diffeomorphism is generated by generic B-fields with arbitrary $H = dB$ although we don't know yet how to construct an emergent gravity theory for such a general B-field.

Finally, though it is claimed that emergent gravity picture is quite general and can go beyond Kähler manifolds, we have not succeeded yet to construct emergent gravity approach for homogeneous non-Kähler manifolds. Of late authors of [49] have formulated an infinitesimal version of T duality using Lie algebraic construction on homogeneous compact manifolds admitting natural torus bundle structure. In their construction it is possible under certain restrictions to find the T-dual of a nilmanifold with non trivial 3-form H flux. They have used the machinery developed by Gualtieri-Cavalcanti [11] to transport generalized complex branes which are a particular kind of submanifold generalizing holomorphic or coisotropic submanifolds of complex or symplectic manifolds in an invariant manner. It would be interesting to study emergent gravity theory on such homogeneous compact manifolds associated to nilpotent Lie groups.

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A T-dual Generalized metric for Torus fibration

We will consider a spacetime M which is a \mathbb{T}^2 -fibration over a arbitrary base manifold B , to find the T -dual generalized metric we first decompose, according to the direct sum $T^*M \cong T^*B \oplus \langle a_1 \rangle \oplus \langle a_2 \rangle$ and $TM \cong TB \oplus \langle \partial_1 \rangle \oplus \langle \partial_2 \rangle$ given by the connection $a = (a_1, a_2)$, the background metric g and the 2-form b

$$\begin{aligned} g &= g_{0,11}a_1 \otimes a_1 + g_{0,22}a_2 \otimes a_2 + g_{0,12}a_1 \otimes a_2 + g_{1,1} \otimes a_1 + g_{1,2} \otimes a_2 + g_2 \\ b &= b_{0,12}a_1 \wedge a_2 + b_{1,1} \wedge a_1 + b_{1,2} \wedge a_2 + b_2 \end{aligned} \quad (\text{A.1})$$

The generalized metric is characterized by the graph of $(g + b)$ denoted by C_+ (See section 3). In the mentioned decomposition of TM , described by the connection a general element of the tangent bundle is given by $X + f_1\partial_1 + f_2\partial_2$. Thus the graph of $(g + b)$ is given by

$$\begin{aligned} &X + f_1\partial_1 \\ &+ f_2\partial_2 \\ &+ \iota_X g_2 + g_{1,1}f_1 + g_{1,2}f_2 + \iota_X b_2 - b_{1,1}f_1 - b_{1,2}f_2 \\ &+ [g_{0,11}f_1 + g_{0,12}f_2 + g_{1,1}(X) - b_0f_2 + b_{1,1}(X)] a_1 \\ &+ [g_{0,22}f_2 + g_{0,12}f_1 + g_{1,2}(X) + b_0f_1 + b_{1,2}(X)] a_2. \end{aligned} \quad (\text{A.2})$$

Under the Courant algebroid isomorphism implementing T-duality such elements of the graph are mapped to elements of $\varphi(C_+)$ of the form:

$$\begin{aligned} &X + [g_{0,11}f_1 + g_{0,12}f_2 + g_{1,1}(X) - b_0f_2 + b_{1,1}(X)] \tilde{\partial}_1 \\ &+ [g_{0,22}f_2 + g_{0,12}f_1 + g_{1,2}(X) + b_0f_1 + b_{1,2}(X)] \tilde{\partial}_2 \\ &+ \iota_X g_2 + g_{1,1}f_1 + g_{1,2}f_2 + \iota_X b_2 - b_{1,1}f_1 - b_{1,2}f_2 \\ &+ f_1 \tilde{a}_1 \\ &+ f_2 \tilde{a}_2. \end{aligned} \quad (\text{A.3})$$

We wish to find a generalized metric $(\tilde{g} + \tilde{b})$ such that the graph \tilde{C}_+ correspond to the elements defined by $\varphi(C_+)$, i.e elements of the form of equation (A.3). Again we consider a decomposition of $(\tilde{g} + \tilde{b})$ with respect to the decomposition $T^*M = T^*B \oplus \langle a_1 \rangle \oplus \langle a_2 \rangle$. With such a decomposition a general element of the graph, of the unknown, $(\tilde{g} + \tilde{b})$ is:

$$\begin{aligned} &X + \tilde{f}_1\tilde{\partial}_1 \\ &+ \tilde{f}_2\tilde{\partial}_2 \\ &+ \iota_X \tilde{g}_2 + \widetilde{g_{1,1}}\tilde{f}_1 + \widetilde{g_{1,2}}\tilde{f}_2 + \iota_X \tilde{b}_2 - \widetilde{b_{1,1}}\tilde{f}_1 - \widetilde{b_{1,2}}\tilde{f}_2 \\ &+ \left[\widetilde{g_{0,11}}\tilde{f}_1 + \widetilde{g_{0,12}}\tilde{f}_2 + \widetilde{g_{1,1}}(X) - \tilde{b}_0\tilde{f}_2 + \widetilde{b_{1,1}}(X) \right] \tilde{a}_1 \\ &+ \left[\widetilde{g_{0,22}}\tilde{f}_2 + \widetilde{g_{0,12}}\tilde{f}_1 + \widetilde{g_{1,2}}(X) + \tilde{b}_0\tilde{f}_1 + \widetilde{b_{1,2}}(X) \right] \tilde{a}_2. \end{aligned} \quad (\text{A.4})$$

Taking $\tilde{f}_1 = g_{0,11}f_1 + g_{0,12}f_2 + g_{1,1}(X) - b_0f_2 + b_{1,1}(X)$ and $\tilde{f}_2 = g_{0,22}f_1 + g_{0,12}f_1 + g_{1,2}(X) + b_0f_1 + b_{1,2}(X)$, the ∂_1 and ∂_2 components agree (notice $X + \tilde{f}_1\partial_1 + \tilde{f}_2\partial_2$ is still a general element of TM). The condition for \tilde{C}_+ to be the graph of $(\tilde{g} + \tilde{b})$ implies an equality of the expressions of (A.3) and (A.4), which in turn sets a system of equations that can be used to find the explicit description of $(\tilde{g} + \tilde{b})$.

Equation arising from the \tilde{a}_1 components and coefficients of f_1 and f_2 , respectively, are:

$$1 = \widetilde{g_{0,11}} [g_{0,11}] + \widetilde{g_{0,12}} [g_{0,12} + b_0] - \tilde{b}_0 [g_{0,12} + b_0],$$

$$0 = \widetilde{g_{0,11}} [g_{0,12} - b_0] + \widetilde{g_{0,12}} [g_{0,22}] - \tilde{b}_0 [g_{0,22}].$$

Using the previous equations we get:

$$\widetilde{g_{0,11}} = \frac{g_{0,22}}{g_{0,11}g_{0,22} - g_{0,12}^2 + b_0^2}; \quad \widetilde{g_{0,22}} = \frac{g_{0,11}}{g_{0,11}g_{0,22} - g_{0,12}^2 + b_0^2}, \quad (\text{A.5})$$

and

$$\widetilde{g_{0,12}} - \widetilde{b_0} = \frac{g_{0,12} - b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}. \quad (\text{A.6})$$

On the other hand, the equation arising from the $\widetilde{a_2}$ components and coefficients of f_1 and f_2 , respectively, are:

$$0 = \widetilde{g_{0,22}} [g_{0,12} + b_0] + \widetilde{g_{0,12}} [g_{0,11}] + \widetilde{b_0} [g_{0,11}],$$

$$1 = \widetilde{g_{0,22}} [g_{0,22}] + \widetilde{g_{0,12}} [g_{0,12} - b_0] + \widetilde{b_0} [g_{0,12} - b_0].$$

Using equation (A.5) and the previous equations, one obtains:

$$\widetilde{g_{0,12}} = \frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}, \quad \widetilde{b_0} = \frac{b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}. \quad (\text{A.7})$$

Now, the 1-form contracted with X from the a_1 and a_2 components are, respectively:

$$0 = \widetilde{g_{0,11}} [g_{1,1}(X) + b_{1,1}(X)] + \widetilde{g_{0,12}} [g_{1,2}(X) + b_{1,2}(X)] + \widetilde{g_{1,1}}(X) - \widetilde{b_0} [g_{1,2}(X) + b_{1,2}(X)] + \widetilde{b_{1,1}}(X),$$

and

$$0 = \widetilde{g_{0,22}} [g_{1,2}(X) + b_{1,2}(X)] + \widetilde{g_{0,12}} [g_{1,1}(X) + b_{1,1}(X)] + \widetilde{g_{1,2}}(X) + \widetilde{b_0} [g_{1,1}(X) + b_{1,1}(X)] + \widetilde{b_{1,2}}(X).$$

Using (A.5), we get

$$\widetilde{g_{1,1}}(X) + \widetilde{b_{1,1}}(X) = \frac{g_{0,22}(g_{1,1} + b_{1,1}) - (g_{0,12} - b_0)(g_{1,2} + b_{1,2})}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}.$$

Furthermore, the f_1 and f_2 coefficients from the T^*B components are, respectively:

$$g_{1,2} - b_{1,2} = [\widetilde{g_{1,1}} - \widetilde{b_{1,1}}] [g_{0,12} - b_0] + [\widetilde{g_{1,2}} - \widetilde{b_{1,2}}] [g_{0,22}],$$

and

$$g_{1,1} - b_{1,1} = [\widetilde{g_{1,1}} - \widetilde{b_{1,1}}] [g_{0,11}] + [\widetilde{g_{1,2}} - \widetilde{b_{1,2}}] [g_{0,12} + b_0].$$

From the previous equations one gets:

$$\widetilde{g_{1,1}}(X) - \widetilde{b_{1,1}}(X) = \frac{(g_{0,12} + b_0)(g_{1,2} - b_{1,2}) - g_{0,22}(g_{1,1} - b_{1,1})}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}, \quad (\text{A.8})$$

and

$$\widetilde{g_{1,2}}(X) - \widetilde{b_{1,2}}(X) = \frac{(g_{0,12} - b_0)(g_{1,1} - b_{1,1}) - g_{0,11}(g_{1,2} - b_{1,2})}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}}. \quad (\text{A.9})$$

Using (A.6), (A.8) and (A.9), we obtain:

$$\begin{aligned} \widetilde{g_{1,1}}(X) = 2 & \left[\frac{-b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,2}(X) \right. \\ & - \frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,2}(X) \\ & \left. + \frac{g_{0,22}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,1}(X) \right], \end{aligned} \quad (\text{A.10})$$

and

$$\begin{aligned} \widetilde{b}_{1,1}(X) = 2 & \left[\frac{b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,2}(X) \right. \\ & - \frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,2}(X) \\ & \left. + \frac{g_{0,22}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,1}(X) \right]. \end{aligned} \quad (\text{A.11})$$

Symmetrically we may obtain:

$$\begin{aligned} \widetilde{g}_{1,2}(X) = 2 & \left[\frac{-b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,1}(X) \right. \\ & - \frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,1}(X) \\ & \left. + \frac{g_{0,11}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,2}(X) \right], \end{aligned} \quad (\text{A.12})$$

and

$$\begin{aligned} \widetilde{b}_{1,2}(X) = 2 & \left[\frac{b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} g_{1,1}(X) \right. \\ & - \frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,1}(X) \\ & \left. + \frac{g_{0,11}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} b_{1,1}(X) \right]. \end{aligned} \quad (\text{A.13})$$

At last the equation obtained from the form contracted with X from the T^*B component is:

$$\iota_X [g_2 + b_2] = \iota_X (\widetilde{g}_2 + \widetilde{b}_2) + [\widetilde{g}_{1,1} - \widetilde{b}_{1,1}] \iota_X [g_{1,1} + b_{1,1}] + [\widetilde{g}_{1,2} - \widetilde{b}_{1,2}] \iota_X [g_{1,2} + b_{1,2}].$$

Using (A.5) - (A.9) and distinguishing the symmetric and anti-symmetric components of $(\widetilde{g} + \widetilde{b})$ we arrive at:

$$\begin{aligned} \widetilde{g}_2 = g_2 + & \left[\frac{b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \otimes b_{1,2} - g_{1,2} \otimes b_{1,1}] \\ & + \left[\frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \otimes g_{1,2} - b_{1,1} \otimes b_{1,2}] \\ & - \left[\frac{g_{0,11}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,2} \otimes g_{1,2} - b_{1,2} \otimes b_{1,2}] \\ & - \left[\frac{g_{0,22}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \otimes g_{1,1} - b_{1,1} \otimes b_{1,2}] \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \widetilde{b}_2 = b_2 + & \left[\frac{b_0}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \wedge g_{1,2} - b_{1,1} \wedge b_{1,2}] \\ & \left[\frac{g_{0,12}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \wedge b_{1,2} - g_{1,1} \wedge b_{1,2}] \\ & + \left[\frac{g_{0,11}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,2} \wedge b_{1,2}] \\ & + \left[\frac{g_{0,22}}{g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22}} \right] [g_{1,1} \wedge b_{1,1}] \end{aligned} \quad (\text{A.15})$$

At last we summarize the results. The dual generalized emergent metric ($\tilde{g} + \tilde{b}$) has a decomposition given by

$$\begin{aligned}\tilde{g} &= \tilde{g}_{0,11}a_1 \otimes a_1 + \tilde{g}_{0,22}a_2 \otimes a_2 + \tilde{g}_{0,12}a_1 \otimes a_2 + \tilde{g}_{1,1} \otimes a_1 + \tilde{g}_{1,2} \otimes a_2 + \tilde{g}_2 \\ \tilde{b} &= \tilde{b}_{0,12}a_1 \wedge a_2 + \tilde{b}_{1,1} \wedge a_1 + \tilde{b}_{1,2} \wedge a_2 + \tilde{b}_2\end{aligned}\quad (\text{A.16})$$

where defining $k = (g_{0,12}^2 - b_0^2 - g_{0,11}g_{0,22})$ the components are given by the following equations:

$$\widetilde{g_{0,12}} = \frac{g_{0,12}}{k}, \quad \tilde{b}_0 = \frac{b_0}{k}. \quad (\text{A.17})$$

$$\widetilde{g_{0,11}} = \frac{-g_{0,22}}{k}; \quad \widetilde{g_{0,22}} = \frac{-g_{0,11}}{k}, \quad (\text{A.18})$$

$$\widetilde{g_{0,12}} - \tilde{b}_0 = \frac{g_{0,12} - b_0}{k}. \quad (\text{A.19})$$

$$\widetilde{g_{1,1}}(X) = \frac{2}{k} [-b_0 b_{1,2}(X) - g_{0,12} g_{1,2}(X) + g_{0,22} g_{1,1}(X)], \quad (\text{A.20})$$

$$\widetilde{g_{1,2}}(X) = \frac{2}{k} [-b_0 b_{1,1}(X) - g_{0,12} g_{1,1}(X) + g_{0,11} g_{1,2}(X)], \quad (\text{A.21})$$

$$\widetilde{b_{1,1}}(X) = \frac{2}{k} [b_0 g_{1,2}(X) - g_{0,12} b_{1,2}(X) + g_{0,22} b_{1,1}(X)]. \quad (\text{A.22})$$

$$\widetilde{b_{1,2}}(X) = \frac{2}{k} [b_0 g_{1,1}(X) - g_{0,12} b_{1,1}(X) + g_{0,11} b_{1,2}(X)], \quad (\text{A.23})$$

$$\begin{aligned}\tilde{g}_2 &= g_2 + \frac{1}{k} [b_0(g_{1,1} \otimes b_{1,2} - g_{1,2} \otimes b_{1,1}) + g_{0,12}(g_{1,1} \otimes g_{1,2} - b_{1,1} \otimes b_{1,2}) \\ &\quad - g_{0,11}(g_{1,2} \otimes g_{1,2} - b_{1,2} \otimes b_{1,2}) - g_{0,22}(g_{1,1} \otimes g_{1,1} - b_{1,1} \otimes b_{1,2})]\end{aligned}\quad (\text{A.24})$$

$$\begin{aligned}\tilde{b}_2 &= b_2 + \frac{1}{k} [b_0(g_{1,1} \wedge g_{1,2} - b_{1,1} \wedge b_{1,2}) + g_{0,12}(g_{1,1} \wedge b_{1,2} - g_{1,1} \wedge b_{1,2}) \\ &\quad + [g_{0,11}(g_{1,2} \wedge b_{1,2}) + g_{0,22}(g_{1,1} \wedge b_{1,1})]\end{aligned}\quad (\text{A.25})$$

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