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# Computational Framework Concerning the Formulation of Maximum Work Principle used in Plasticity, Materials Forming and Tribology as a Consequence of a Variational Optimization Problem Defined From the Constructal Law

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**Abstract.** This research work proposes a fundamental application of the Constructal Theory developed by Prof. Adrian BEJAN of Duke University (USA) to prove on a mathematical point of view the “Principle” of Maximum Work, used by the theory of plasticity, material forming and tribology, as a consequence of the global solution defining a constrained variational optimization problem. According to the first and second law of thermodynamics, the principal law of Constructal Theory try to complete them with a quantitative prediction of the natural tendency of any finite size system to evolve towards an optimal space-time configuration minimizing the losses and the entropy generation simultaneously with a required maximum of the global entropy. In this sense, regarding a material plastic deformation characterising the forming processes, among all possible and admissible flow undergoing well-specified boundary conditions and loadings, the real one is the one who minimizes the sum of the dissipated power of volume deformation and surfaces friction. Thus, all the mechanical variables defining the real mechanical state (velocities, stresses, strain and strain rate) are those ones which minimize the total dissipated power. A variational minimization problem under a lot of defined constraints is then obtained. Using the Principle of Virtual Powers it can be shown finally that the “Principle” of Maximum Work, used particularly in metals plasticity, is obtained as a consequence of a minimization problem under constraints based on the Constructal Theory. This generalizes its application to any type of continuous media (metals, polymers, fluid, mushy state) and allows proving an equivalent form for the friction stresses occurring on contact interfaces. The convexity properties of both the plastic and the friction potential together with their normal rule properties can be also proven using the proposed mathematical framework. It is concluded that only the rheological and tribological flow laws associated with a potential become to satisfy the second thermodynamics principle completed with the Constructal Law. Analytical computations concerning plane and cylindrical crushing show the feasibility of the proposed minimization problem formulation to define material flow giving accurate approximate solutions. In order to valid the whole presented theory, comparisons are made using classical analyses based on the upper and lower bound theorems (obtained as consequences of the proposed optimization variational problem), the well-known slices method and a finite element modelling (FEM). A second application concerning the anisotropic formulation of a Coulomb friction law from a quadratic convex tribologic potential will be presented to define contact evolution during rectilinear sliding of a circular pion on a plane laminated thick plate surface along different orientations.



## 1. Introduction

During the last twenty years Prof. Adrian BEJAN of Duke University has developed theoretical and applied researches concerning the thermodynamic optimization of complex systems based on the principle called as constructal defining any system evolution by the maximization of entropy, minimization of generated entropy and minimization of resistance and losses related to local or global flows searching to maximize the speed to reach a stable equilibrium state. As mentioned in previous scientific works [1-2], the Constructal Theory of Professor A. BEJAN can be seen as a definition and resolution of a general problem of optimization under constraints taking into account the postulate than: *“for a finite-size system to persists in time (to live or to be able to survive), it must evolve in such a way that it provides easier access to the imposed (global) currents that flow through it”* as to facilitate access as much as possible under the constraints flows or which cross them minimizing the corresponding losses [1]. In addition, it is also observed that all system search to optimize the distribution of their imperfections in order to facilitate the flow and to minimize the local resistance or the required transformation powers. Numerous studies concerning thermal problems, fluid or porous media flow, nature, economic or societal behavior confirm this principle [2]. Starting from recent author’s scientific works, this article proposes to consolidate a general mathematical proof of the "Principle" of Maximum Work used in materials plasticity through its definition as a theorem which results from the application of the Constructal Principle seen as being a more general postulate from a thermodynamic point of view and valid for any type of continuum media [3-5]. It is well known that the Maximum Work Principle (MWP) is used by plasticity theory [6] to obtain associated flow laws or to define constitutive equations related to describe the material’s strength and to develop analytical or numerical computations in particular during forming processes. A lot of previous works [7-8] try only to explain and to postulate the Maximum Working Principle by a purely phenomenological approach regarding polycrystalline metals starting from local slips of atomic particles planes having a maximum density. By exploiting the Virtual Powers Principle (VPP) defining the mechanic dynamic equilibrium if is applied the Constructal Theory, the Maximum Work Principle can be obtained as a direct consequence of a general variational optimization problem which search to minimize the losses of material flow, see to minimize the sum of all the dissipations energies. After theoretical backgrounds accompanied by their consequences concerning the convexity and the normality rule law of rheological and tribological potentials describing the bulk material behavior and contact interfaces, will be recalled the theorems of the upper and lower bound together with applications concerning a plane compression and a cylindrical crushing taking into account a rigid-plastic behavior with a Tresca plastic friction law. A validation of the proposed analyzes together with all the obtained analytical solutions will be made through comparisons between the estimation of the compression loads evolution and the corresponding numerical results given by Finite Element Modeling using Forge2®.

## 2. Theoretical framework

According to the theory of continuum mechanics for all material flow in a well-defined body volume  $\Omega$ , the dynamic mechanical equilibrium can be written from the Virtual Powers Principle (VPP):

$$\int_{\Omega} [\sigma] : [\dot{\epsilon}^*] dV + \int_{\partial\Omega} -\bar{\tau} \cdot \Delta \bar{v}^* dS = \int_{\partial\Omega'} \bar{T} \cdot \bar{v}^d dS' + \int_{\partial\Omega''} \bar{T}^d \cdot \bar{v}^* dS'' + \int_{\Omega} \rho \bar{f} \cdot \bar{v}^* dV - \int_{\Omega} \rho \frac{d\bar{v}^*}{dt} \cdot \bar{v}^* dV \quad (1)$$

It is known that this principle, equivalent to Newton's fundamental law, is valid for any field of admissible virtual speeds  $\bar{v}^*$  respecting the boundary conditions taking into account the Cauchy stress tensor  $[\sigma]$ , the virtual strain rate tensor  $[\dot{\epsilon}^*] = \frac{1}{2} \{ [grad(\bar{v}^*)] + [grad(\bar{v}^*)]^T \}$ , the friction stress vector  $\bar{\tau}$  acting on the contact surface or on the surface of velocities discontinuities  $\partial\Omega$ , the specific loading  $\bar{T}$ , the imposed velocities  $\bar{v}^d$  on the body border part  $\partial\Omega'$ , the specific imposed loads  $\bar{T}^d$  on the body border part  $\partial\Omega''$ , the specific mass forces  $\bar{f}$  and the specific density  $\rho$ . For the real velocity field  $\bar{v}$  of material flow the dynamic equilibrium is described by the Power Work Principle (PWP):

$$\int_{\Omega} [\sigma] : [\dot{\epsilon}] dV + \int_{\partial\Omega} -\bar{\tau} \cdot \Delta \bar{v} dS = \int_{\partial\Omega'} \bar{T} \cdot \bar{v}^d dS' + \int_{\partial\Omega''} \bar{T}^d \cdot \bar{v} dS'' + \int_{\Omega} \rho \bar{f} \cdot \bar{v} dV - \int_{\Omega} \rho \frac{d\bar{v}}{dt} \cdot \bar{v} dV \quad (2)$$

### 2.1. Theorem of « Maximal Work Principle »

Using the constructal law, in the case of a deformable material or during a forming process, we can postulate that the flow of the material takes place such that it minimizes the sum of the dissipated powers corresponding to the flow of the material, to the friction at the contact interfaces or velocities discontinuities and to the imposed loads. Consequently, the real values of all the kinematic and mechanical variables (velocities, stresses, strains, strain rates) are those which minimize the total power of dissipation. Thus, for any state of material defined by virtual mechanics variables, the real ones must minimize the functional of total dissipation power. It can be then concluded that, for all plastic materials (metallic or non-metallic, fluids or polymers) and for any virtual state, the real strain rate tensor  $[\dot{\epsilon}]$  and the real Cauchy stress tensor  $[\sigma]$  corresponding to the material flow can be obtained by minimizing the sum of the dissipated plastic power, of the friction power or of the velocities discontinuities and of the imposed loads power. In addition, the plasticity of the material is generally governed in terms of the stress tensor and the friction vector through the definition of a plastic criterion defined by a multi-variables scalar function  $\Phi_p([\sigma])=0$  together with a similar form regarding the friction term i.e.  $\Psi_f(\bar{\tau})=0$ . In this case, from all other admissible virtual speed fields  $v^*$  (different with respect to the real one) characterized by a virtual strain rate tensor  $[\dot{\epsilon}^*] \neq [\dot{\epsilon}]$  and a virtual plastic constraints tensor  $[\sigma^*]$  ( $\Phi_p([\sigma^*])=0$  et  $\Psi_f(\bar{\tau}^*)=0$ ), the real material plastic flow state can be obtained by minimizing the functional defined through the virtual total dissipated power:

$$\dot{W}_d = \text{Min}(\dot{W}_d^*) \text{ with } \dot{W}_d^* = \int_{\Omega} [\sigma^*] : [\dot{\epsilon}^*] dV + \int_{\partial\Omega} -\bar{\tau}^* \cdot \Delta \bar{v}^* dS + \int_{\partial\Omega''} -\bar{T}^d \cdot \bar{v}^* dS'' \quad (3)$$

Using the boundary conditions concerning the kinematics and the loading of the material, a variational minimization problem under constraints is then obtained. As shown by the Constructal Theory the optimization solution is obtained for the real flow state. So for all virtual states it is required to have the following inequality:

$$\int_{\Omega} [\sigma] : [\dot{\epsilon}] dV + \int_{\partial\Omega} -\bar{\tau} \cdot \Delta \bar{v} dS + \int_{\partial\Omega''} -\bar{T}^d \cdot \bar{v} dS'' \leq \int_{\Omega} [\sigma^*] : [\dot{\epsilon}^*] dV + \int_{\partial\Omega} -\bar{\tau}^* \cdot \Delta \bar{v}^* dS + \int_{\partial\Omega''} -\bar{T}^d \cdot \bar{v}^* dS'' \quad (4)$$

For consistent materials and quasi-static conditions, the mass and the inertial forces can be neglected, so the VPP principle can be then written in the following simplified form:

$$\int_{\Omega} [\sigma] : [\dot{\epsilon}] dV + \int_{\partial\Omega} -\bar{\tau} \cdot \Delta \bar{v} dS + \int_{\partial\Omega''} -\bar{T}^d \cdot \bar{v} dS'' = \int_{\Omega} [\sigma] : [\dot{\epsilon}^*] dV + \int_{\partial\Omega} -\bar{\tau} \cdot \Delta \bar{v}^* dS + \int_{\partial\Omega''} -\bar{T}^d \cdot \bar{v}^* dS'' \quad (5)$$

Starting from (4) and (5), it can be concluded that for any virtual stress state  $[\sigma^*]$  ( $\Phi_p([\sigma^*])=0$ ) or any virtual friction  $\bar{\tau}^*$  ( $\Psi_f(\bar{\tau}^*)=0$ ) and for any body  $\Omega$  is obtained the below equivalent form:

$$\int_{\Omega} ([\sigma^*] - [\sigma]) : [\dot{\epsilon}^*] dV + \int_{\partial\Omega'} (\bar{\tau} - \bar{\tau}^*) \cdot \Delta \bar{v}^* dS' \geq 0 \quad (6)$$

This condition requires having positive values for each term i.e.:

$$([\sigma^*] - [\sigma]) : [\dot{\epsilon}^*] \geq 0, \forall [\sigma^*], \Phi_p([\sigma^*])=0 \text{ and } -(\bar{\tau}^* - \bar{\tau}) \cdot \Delta \bar{v}^* \geq 0, \forall \bar{\tau}^*, \Psi_f(\bar{\tau}^*)=0 \quad (7)$$

Indeed if one considers that there exists a virtual state of the stresses or friction for which the terms are negative, one could build another identical virtual state with real state of stresses outside the field for which the terms of the integrals are negative (therefore with zero integrals values outside this domain) and one would obtain a negative value for the inequality (6) which contradicts the positive value to be respected. In the opposite sense, taking into account the real plastic flow characterized by the velocity field  $v$ , the strain rate tensor  $[\dot{\epsilon}]$  and the Cauchy stress tensor  $[\sigma]$ , any other state of the admissible stresses must also verify the following inequalities:

$$([\sigma] - [\sigma^*]) : [\dot{\epsilon}] \geq 0, \forall [\sigma^*], \Phi_p([\sigma^*]) = 0 \text{ and } -(\bar{\tau} - \bar{\tau}^*) \cdot \Delta \bar{v} \geq 0, \forall \bar{\tau}^*, \Psi_f(\bar{\tau}^*) = 0 \quad (8)$$

Starting from the above obtained relationships (7) and (8), it is possible to prove mathematically the convex shape of the potential functions defining the plastic and friction criteria and subsequently the known property of normality to the criterion. Consequently the strain rate should be proportional to the gradient of the plastic criterion with respect to each components of the stress i.e. :

$$[\dot{\epsilon}] = \lambda_p \partial \Phi_p / \partial [\sigma], \lambda_p \geq 0 \quad (9)$$

In the same sense the associated friction law can be written in the form:

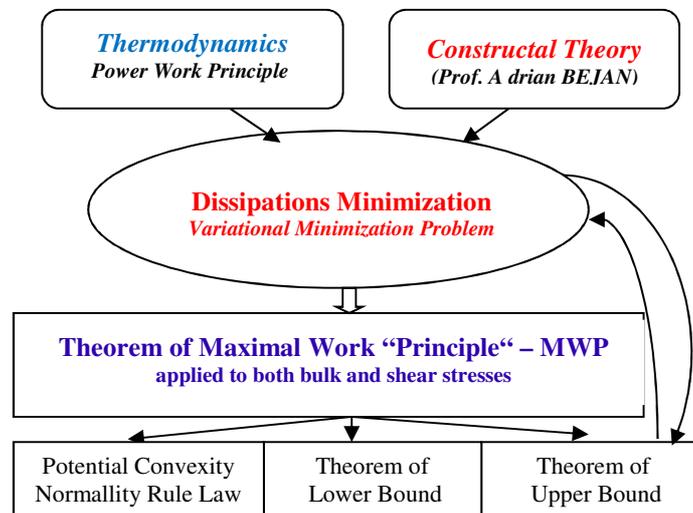
$$\Delta \bar{v} = -\lambda_f \partial \Psi_f / \partial \bar{\tau}, \lambda_f \geq 0 \quad (10)$$

In this case, based on the property of the convexity of plastic criteria  $\Phi_p([\sigma^*])$  and  $\Psi_f(\bar{\tau}^*)$ , the two inequalities expressed by (8) can be extended to the virtual stress and friction shear respecting  $\Phi_p([\sigma^*]) \leq 0$  and  $\Psi_f(\bar{\tau}^*) \leq 0$ . It is then possible to conclude that for any state of virtual stresses is obtained:

$$([\sigma] - [\sigma^*]) : [\dot{\epsilon}] \geq 0, \forall [\sigma^*], \Phi_p([\sigma^*]) \leq 0 \text{ and } -(\bar{\tau} - \bar{\tau}^*) \cdot \Delta \bar{v} \geq 0, \forall \bar{\tau}^*, \Psi_f(\bar{\tau}^*) \leq 0 \quad (11)$$

The first inequality is known in the plasticity theory as the Maximum Work Principle [4-8]. It is proved here that this can be obtained as a consequence of Constructal Theory [1], [3] and of the Principle of Virtual Powers. The second inequality reflects the same principle applied to the friction stress state.

## 2.2. Practical Consequences and Synthesis



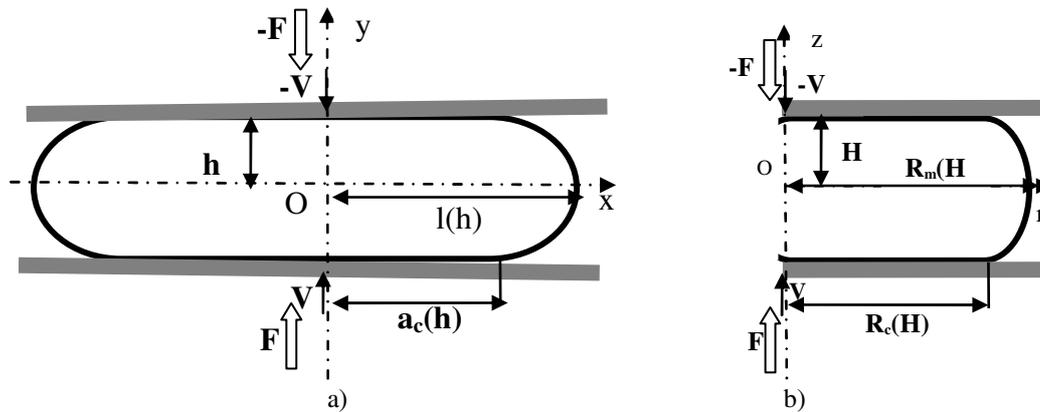
**Figure 1.** Flowchart concerning the formulation of the dissipation variational minimization problem based on the Constructal Theory and the proof of the "Principle" of Maximum Work (MWP) together with the characteristic properties and theorems [3-5]

Using the theoretical proofs presented above, it can be concluded that the Maximum Work Principle can be applied for any type of continuous media: fluid, solid or pasty material, metallic or non-metallic, as well as to define in a similar form the principle of maximum power concerning the friction stresses. Specific expressions concerning the convex criterion of plasticity or of friction  $\Phi_p([\sigma]) = 0$ ,  $\Psi_f(\bar{\tau}) = 0$  can be proposed and used to define the isotropic or anisotropic plastic behavior of materials, respectively of the constitutive laws defining sliding at the contact interfaces. It is also possible to obtain and prove the theorems of the Lower and Upper Bound (Figure 1) [6-8].

These can also be considered as variational optimization formulations and are frequently used to obtain analytical estimations of the loads applied during a material forming process under particular assumptions. They are also used to compare and validate the results obtained by Finite Element Modeling (FEM). Regarding the Upper Bound Theorem [8], it is observed that it is indeed obtained an equivalent form of the variational minimization problem (3). One can thus conclude on the coherence and especially on the equivalence between the Maximum Work Principle and the Principle of Minimization of the total dissipations or losses defined by the Constructal Theory.

### 3. Applications: plane compression and cylindrical crushing

In order to validate the variational minimization principle of dissipations expressed by the optimization problem (3) postulating the real state of a material plastic deformation, two cases of simple forming are considered: plane compression and cylindrical crushing ( Figure 2a and Figure 2b).



**Figure 2.** Simple forming operations [5]: a) plane compression, b) cylindrical crushing

It is assuming a material with a perfect rigid-plastic behaviour defined by a constant value of the equivalent Von-Mises stress  $\sigma_0$  (i.e.  $\Phi_p([\sigma^*]) = \bar{\sigma}^* - \sigma_0 = 0$ ). In view of large plastic deformations generated during a plane compression or crushing process on a double-action press at constant speed  $V$ , a Tresca friction is taken into account on the specimen-tool surface contacts  $\tau = \bar{m}\sigma_0 / \sqrt{3}$  with  $\bar{m} \in [0, 1]$ . Corresponding to the coordinate systems of Figure 2, taking into account the axial, respectively revolution symmetry, for each time instant  $t$  the geometry of the plane compression parallelepiped sample is defined by a transverse surface  $S_t = 2a \times 2h$  and a longitudinal surface  $S_p = 2a \times L$  considering  $a \approx l(h) \approx a_c(h)$ , while in the case of cylindrical crushing the specimen is defined by a cylindrical surface  $S_c$  of radius  $R \approx R_m(H) \approx R_c(H)$  and a height  $2H$ . To be able to exploit the variational principle expressing the real state of the flow through the minimization problem (3), it can be defined a virtual kinematics through an incompressible velocity field ( $div(\vec{v}^*) = 0$ ) and kinematically admissible (in accordance with the boundaries conditions defined in Figure 2) by expressing the velocities components using the below expressions proposed by Avitzur in the case of a plane compression respectively of a cylindrical crushing [9]:

$$\begin{aligned}
 v_x^*(x, y) &= AV \frac{x}{h} \exp(-\beta y / h) & v_r^*(r, z) &= A'V \frac{r}{H} \exp(-\beta z / 2H) \\
 v_y^*(y) &= AV [\exp(-\beta y / h) - 1] / \beta & v_z^*(z) &= 4A'V [\exp(-\beta z / 2H) - 1] / \beta \\
 v_x^*(0, y) &= 0, v_y^*(0) = 0, v_y^*(h) = -V & v_r^*(0, z) &= 0, v_\theta^*(r, z) = 0, v_z^*(0) = 0, v_z^*(H) = -V \\
 A &= \beta / [1 - \exp(-\beta)], x \in [0, a], y \in [0, h] & A' &= \beta / 4 [1 - \exp(-\beta / 2)], r \in [0, R], z \in [0, H]
 \end{aligned} \tag{12}$$

Here  $\beta$  represents a dimensionless parameter varying the shape of the virtual velocity field. Based on the variational problem formulated by (3), the real total dissipations power  $\dot{W}_d$  must verify:

$$\dot{W}_d = \text{Min } \dot{W}_d^* \leq \text{Min } \dot{W}_d^*(\beta) \tag{13}$$

Since we have  $\vec{T}^d = 0$  the subfamily of the virtual total power of the dissipations is defined by:

$$\dot{W}_d^*(\beta) = \int_{\Omega} [\sigma^*] : [\dot{\epsilon}^*] dV + \int_{\partial\Omega} -\vec{\tau}^* \cdot \Delta \vec{v}^* dS = \int_{\Omega} \sigma_0 \dot{\epsilon}^* dV + \int_{\partial\Omega} (\bar{m}\sigma_0 / \sqrt{3}) \cdot |\Delta \vec{v}^*| dS \tag{14}$$

where:

$$\dot{\epsilon}^* = \sqrt{(2/3)\text{Trace}\{[\dot{\epsilon}^*]^2\}} \text{ and } [\dot{\epsilon}^*] = \frac{1}{2}\{[\text{grad}(\vec{v}^*)] + [\text{grad}(\vec{v}^*)]^T\} \tag{15}$$

The summary of the analytical computations of all the terms involved by the evaluation of  $\dot{W}_d^*$  is presented in Table 1 using the assessments detailed by the author in previous works [3-5] on the basis of design principles and material forming theory presented in [7-11]. Given the complexity of analytical evaluation the  $\beta_{opt}$  which minimize  $\dot{W}_d^*(\beta)$ , this study is limited to the necessary condition  $\dot{W}_d \leq \text{Min } \dot{W}_d^*(\beta) \leq \dot{W}_d^*(0)$ .

**Table 1.** Definition of the slenderness factor  $e$ , of the shape functions with respect to the dimensionless kinematic parameter  $\beta$ , of the total reference power  $P$  and of the functional expressing the virtual power of dissipations  $\dot{W}_d^*$  [5]

	Plane Compression	Cylindrical Crushing
$e$ (slenderness)	$a/h$	$R/H$
$V'$ (volume)	$2ahL$	$\pi R^2 H$
$S_p, S_c$ (area)	$2aL$	$\pi R^2$
$\alpha(\beta)$	$\beta e / 2$	$ \beta  e / 2$
	$\beta \rightarrow 0$	$0$
$\phi[\alpha(\beta)]$	$\beta \neq 0$ $\sqrt{1 + \alpha(\beta)^2} + \frac{\ln[\alpha(\beta) + \sqrt{1 + \alpha(\beta)^2}]}{\alpha(\beta)}$	$8\alpha(\beta) \left[ \left( \frac{1}{12} + \alpha(\beta)^{-2} \right)^{3/2} - \left( \alpha(\beta)^{-2} \right)^{3/2} \right]$
	$\beta \rightarrow 0$	$2$
$\varphi(\beta)$	$\beta \neq 0$ $\beta \exp(-\beta) / [1 - \exp(-\beta)]$	$2(\beta/2) \exp(-\beta/2) / [1 - \exp(-\beta/2)]$
	$\beta \rightarrow 0$	$1$
	Condition nécessaire	Condition nécessaire
$\beta_{opt}$	$2 \frac{d\phi}{d\beta} + \bar{m}e \frac{d\varphi}{d\beta} = 0 \Leftrightarrow \frac{d\phi}{d\alpha} + \bar{m} \frac{d\varphi}{d\beta} = 0$	$2 \frac{d\phi}{d\beta} + \frac{\bar{m}e}{3\sqrt{3}} \frac{d\varphi}{d\beta} = 0 \Leftrightarrow \frac{\beta}{ \beta } \frac{d\phi}{d\alpha} + \frac{\bar{m}}{3\sqrt{3}} \frac{d\varphi}{d\beta} = 0$
$P$ (Ref. Power)	$\frac{4}{\sqrt{3}} \sigma_0 a L V = \frac{2}{\sqrt{3}} \sigma_0 S_p V'$	$\sigma_0 \pi R^2 V = \sigma_0 S_c V'$
$\dot{W}_d^*(\beta)$	$P[\phi[\alpha(\beta)] + \bar{m}e\varphi(\beta)/2]$	$P[2\phi[\alpha(\beta)] + \bar{m}e\varphi(\beta)/3\sqrt{3}]$
$\dot{W}_d^*(0)$	$\dot{W}_d^*(0) = 2P \left[ 1 + \bar{m} \frac{e}{4} \right]$	$\dot{W}_d^*(0) = 2P \left[ 1 + \bar{m} \frac{e}{3\sqrt{3}} \right]$

Thus concerning the plane compression the real load  $F$  must verify:

$$F \leq \tilde{F} = \frac{\dot{W}_d^*(0)}{2V} = \frac{4aL\sigma_0}{\sqrt{3}} [1 + \bar{m}e/4] = \frac{2\sigma_0 S_p}{\sqrt{3}} [1 + \bar{m}e/4] \tag{16}$$

Starting from the lower bound theorem, using a specific statically admissible virtual stress state  $\text{Div}([\sigma]^*) = 0$  [3], while relying on the virtual power principle and on the MWP as a consequence of the minimization problem (3), it can be written:

$$F \geq \tilde{F} = \frac{4aL\sigma_0}{\sqrt{3}} \left[ \sqrt{1-\bar{m}^2} + \bar{m}e/4 \right] = \frac{2\sigma_0 S_p}{\sqrt{3}} \left[ \sqrt{1-\bar{m}^2} + \bar{m}e/4 \right] \quad (17)$$

It is easy to see that the real compressive load could be approximated by the mean value  $F \approx (\tilde{F} + \tilde{\tilde{F}})/2$  with a maximal estimation error of  $(1 - \sqrt{1 - \bar{m}^2}) / (1 + \bar{m}e/4)$  with  $\bar{m} \in [0, 1]$ .

In the particular case of a compression without friction ( $\bar{m} = 0$ )  $\tilde{F} = \tilde{\tilde{F}}$  and  $F = \frac{4aL\sigma_0}{\sqrt{3}} = \frac{2}{\sqrt{3}}\sigma_0 S_p$ .

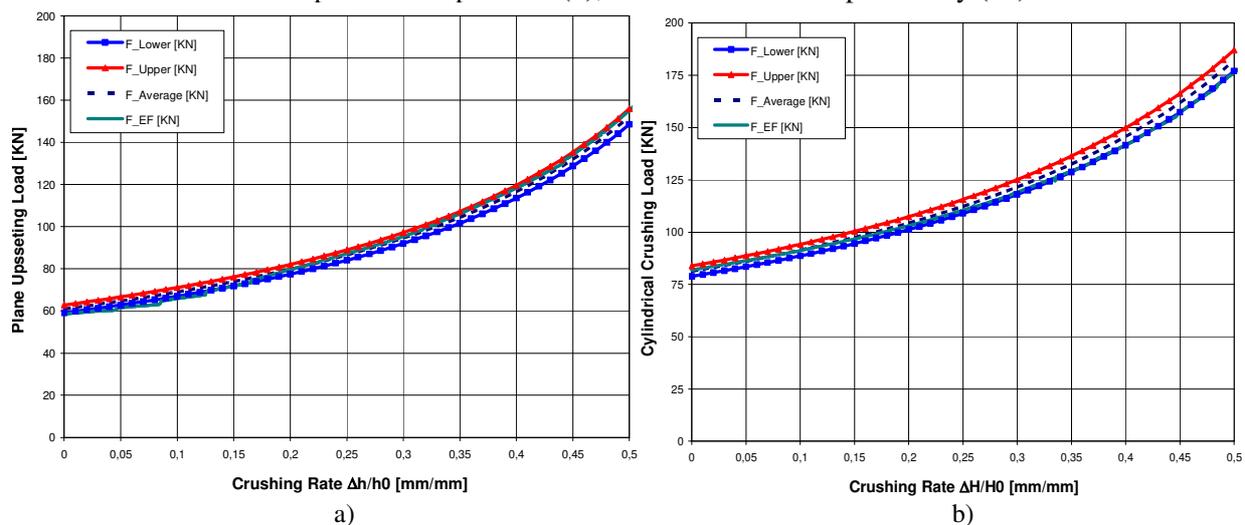
This expression is the same with that obtained by an analytical study based on the resolution of the differential incompressibility equation (defining the real velocities by  $v_x(x) = Vx/h, v_y(y) = -Vy/h$ ), on the static equilibrium equation and of the law of normality of the potential rigid-plastic (giving the stresses defined by  $\sigma_{xx} = 0, \sigma_{yy} = -2\sigma_0/\sqrt{3}, \sigma_{zz} = -\sigma_0/\sqrt{3}$ ) [6]. It is important to note that in this case, by choosing  $\beta = 0$  (because as can be seen in the Table 1  $\beta_{opt} = 0$  if  $\bar{m} = 0$ ), one finds practically the overall minimum of the expression (13), which proves well that the real flow is the solution of the variational optimization problem (3). A similar analysis for the cylindrical crushing gives:

$$F \leq \tilde{F} = \frac{\dot{W}_d^*(0)}{2V} = \pi R^2 \sigma_0 \left[ 1 + \bar{m}e/3\sqrt{3} \right] = \sigma_0 S_c \left[ 1 + \bar{m}e/3\sqrt{3} \right] \quad (18)$$

Starting from the detailed results in [4] it is obtained:

$$F \geq \tilde{F} = \pi R^2 \sigma_0 \left[ \sqrt{1-\bar{m}^2} + \bar{m}e/3\sqrt{3} \right] = \sigma_0 S_c \left[ \sqrt{1-\bar{m}^2} + \bar{m}e/3\sqrt{3} \right] \quad (19)$$

The average crushing load may be defined by  $F \approx (\tilde{F} + \tilde{\tilde{F}})/2$  with a maximal estimation error around  $(1 - \sqrt{1 - \bar{m}^2}) / (1 + \bar{m}e/3\sqrt{3})$ . For a crushing without friction ( $\bar{m} = 0$ )  $F = \tilde{F} = \tilde{\tilde{F}} = \pi R^2 \sigma_0$  i.e.  $F = \sigma_0 S_c$ . Similarly, it is found that this coincides with the analytical solution (real velocity defined by  $v_r(r) = Vr/H, v_z(z) = -Vz/H$  and diagonal stress tensor  $\sigma_{rr} = 0, \sigma_{\theta\theta} = 0, \sigma_{zz} = -\sigma_0$ ) [6]. The same conclusion is reached i.e. we have practically  $\beta_{opt} = 0$  what again justifies the achievement of the overall minimum of the optimization problem (3), see the condition expressed by (13).



**Figure 3.** Comparisons of the compression load-crushing rate evolutions obtained by analytical and numerical estimations: a) plane compression, b) cylindrical crushing

To illustrate the degree of accuracy of the analytical estimates of the forming forces compared to a Finite Element (EF) modeling using the Forge2® software, a parallelepiped sample of dimensions 20 mm x 20 mm x 20 mm with an initial slenderness  $e_0 = 1$  and a cylindrical specimen of equivalent

volume defined by a radius of 10 mm, a height of 20 mm and the same initial slenderness  $e_0 = 1$ . Assuming  $\sigma_0 = 250$  MPa and an average friction defined by  $\bar{m} = 0.35$  the load-stroke evolution curves are shown in Figure 3a and Figure 3b. It is easy to see that in both cases (plane compression and cylindrical crushing) the EF solution of the evolution of the load is well between the upper  $\tilde{F}$  and lower  $\tilde{\tilde{F}}$  theoretical limits with values very close to those given by the estimated average  $F \approx (\tilde{F} + \tilde{\tilde{F}}) / 2$  (error less than 1%) while the load estimation error is maximum 6.5% for a crushing rate of 0,5 with a logarithmic deformation of about 70%, seen by the difference of the variation between the lower and upper limit.

In the ideal case without friction the analytical curves are superposed and have values practically equal to the numerical finite elements ones. Furthermore, for an identical initial slenderness and an almost equivalent material volume (with a ratio of  $4/\pi$ ) the cylindrical crushing force have values approximately 1.15-1.25 times greater than in the case of a plane compression, thus regaining the ratio  $2/\sqrt{3}$  between the two modes of deformation.

#### 4. Conclusions

During the content of this article, it was proved that for a plastic flow of all continuous media, starting from the general principle defined by the Constructal Theory concerning the minimization of the total dissipated energy characterizing the evolution of a finite size system, in particular a deformable material body, one obtains and one proven in a more general form the “Principle” of Maximum Work both with regard to the state of material bulk stresses and that of friction ones. The application for a plane compression and a cylindrical crushing of a rigid-plastic material shows the feasibility of the proposed formulation through the solve of a corresponding variational minimization problem expressed in terms of total dissipated energy, with possibilities to obtain analytical estimations of the required material forming loads evolution. Thus the expressions and the optimal analytical values of the upper and lower limits corresponding to the evolution of the compression load must be seen as a consequence of the proposed variational optimization problem obtained as a consequence of the Constructal Theory. The comparisons with the simulation results obtained by finite element modeling show the feasibility and especially the high degree precision of obtained analytical values.

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