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GRADIENT ESTIMATES AND MAXIMAL DISSIPATIVITY FOR THE KOLMOGOROV OPERATOR IN Φ^4_2

GIUSEPPE DA PRATO AND ARNAUD DEBUSSCHE

ABSTRACT. We consider the transition semigroup P_t of the Φ^4_2 stochastic quantisation on the torus $\mathbb{T}^2$ and prove the following new estimate (Theorem 3.9)

$$|DP_t\varphi(x)\cdot h| \le c\, t^{-\beta}\, |h|_{C^{-s}} \|\varphi\|_0\, (1+|x|_{C^{-\alpha}})^{\gamma},$$

for some α, β, γ, s positive. Thanks to this estimate, we show that cylindrical functions are a core for the corresponding Kolmogorov equation. Some consequences of this fact are discussed in a final remark.

1. INTRODUCTION

We consider the equation of the Φ^4_2 stochastic quantisation on the torus $\mathbb{T}^2$:

$$\begin{cases} dX = (AX - :X^3:)dt + dW, \\ \\ X(0) = x, \end{cases} \tag{1.1}$$

where

$$Ax = \Delta x - x, \quad D(A) = H^2(\mathbb{T}^2).$$

This equation has been the object of several studies. In particular, in [DaDe03-2] existence and uniqueness of a mild solution to (1.1) has been proved for almost all initial datas x with respect to the Gibbs measure ν defined below; later in [MoWe15] well–posedness was proved *for all* x in a negative Besov space. Moreover, in [HaMa16] the strong Feller property of the corresponding transition semigroup $(P_t)_{t\ge 0}$ was proved. This latter result is improved in [TsWe18] where it is proved that, for $t > 0$, P_t maps bounded Borel functions to α-Hölder functions for some $\alpha \in (0, 1)$.

The main result of the present paper (Theorem 3.9 below) goes further in this direction: we prove that P_t maps bounded borelian functions to Lipschitz functions and give an estimate of the gradient of $P_t\varphi$. We use a method similar to the one used in [DaDe03-1] and [DaDe07] as well as similar estimates as in [TsWe18]. In the second part we use this result for showing that cylindrical functions are a core for the infinitesimal generator $\mathcal{K}$ of P_t and present some consequences of this fact.

2. NOTATIONS

We use the classical $L^p = L^p(\mathbb{T}^2)$ spaces for $p \in [1, \infty]$. For $p = 2$, we write $H = L^2$. We use also the L^2 based Sobolev spaces $H^s = H^s(\mathbb{T}^2)$ for $s \in \mathbb{R}$ and the Besov spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{T}^2)$, for $s \in \mathbb{R}$, $p, q \in [1, \infty]$. These are defined in terms of Fourier series and Littlewood-Paley theory (see [BaChDa11]). Note that $H^s = B^s_{2,2}$. When $p = q = \infty$, we write $B^s_{\infty,\infty} = C^s$ and for $s > 0$ non integer these spaces coincide with the classical Hölder spaces.

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Besov spaces are convenient when multiplying two functions when one has negative regularity. Recall that for $p_1,\, p_2 \in [1,\infty]$ and $\alpha_1,\, \alpha_2 \in \mathbb{R}$ with $\alpha_1+\alpha_2 > 0$ we have for $\alpha = \alpha_1 \wedge \alpha_2 \wedge (\alpha_1+\alpha_2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\tilde{p} = p_1 \vee p_2$

$$\|uv\|_{B^{\alpha}_{p,\tilde{p}}} \leq C \|u\|_{B^{\alpha_1}_{p_1,p_1}} \|v\|_{B^{\alpha_2}_{p_2,p_2}}$$

and for any $\kappa > 0$

$$\|uv\|_{B^{\alpha-\kappa}_{p,p}} \leq C \|u\|_{B^{\alpha_1}_{p_1,p_1}} \|v\|_{B^{\alpha_2}_{p_2,p_2}}\,.$$

We use the eigenprojections Π_N corresponding to the eigenvalues $k^2 \leq N$, $k \in \mathbb{Z}^2$, of A.

Define the Gaussian measure

$$\mu = \prod_{k\in\mathbb{Z}^2} N\left(0, \frac{1}{1+|k|^2}\right),\ \text{on}\ \mathcal{H} = \mathbb{R}^{\mathbb{Z}^2},$$

and for $x_N \in \Pi_N H$

$$:x_N^2: = x_N^2 - \rho_N^2, \quad :x_N^3 := x_N^3 - 3\rho_N^2 x_N,$$

where

$$\rho_N = \frac{1}{2\pi}\left[\sum_{|k|\leq N} \frac{1}{1+|k|^2}\right]^{1/2}.$$

More generally, the n$^{\text{th}}$ Wick power of x_N is defined by $:x_N^n := \sqrt{n!}\rho_N^n H_n(\frac{1}{\rho_N}x_N)$ where H_n is the n$^{\text{th}}$ Hermite polynomial.

As well known, there exists the limit

$$\lim_{N\to\infty} :(\Pi_N x)^n: =: x^n :,\ \text{in}\ L^r(\mathcal{H},\mu;B^s_{p,q}), \tag{2.1}$$

for any $r \in [1,\infty),\ p,q \in [1,\infty],\ s<0$.

Then, we define

$$Z_\infty(t) = \int_{-\infty}^t e^{(t-s)A} dW(s).$$

Since $Z_\infty(t)$ has law μ, we know that $:\Pi_N Z_\infty^n:$ converges to $:Z_\infty^n:$ in $L^r(\mathcal{H},\mu;B^s_{p,q})$ for any $r \in [1,\infty),\ p,q \in [1,\infty],\ s<0$

We set moreover $X = Y_\infty + Z_\infty = Y + Z$ where

$$Z(t) = \int_0^t e^{(t-s)A} dW(s) = Z_\infty(t) - e^{At} Z_\infty(0),$$

Clearly:

$$:(\Pi_N X)^3: = (\Pi_N Y_\infty)^3 + 3(\Pi_N Y_\infty)^2 \Pi_N Z_\infty + 3(\Pi_N Y_\infty) :(\Pi_N Z_\infty)^2: + :(\Pi_N Z_\infty)^3:.$$

It is therefore natural to consider the following definition for $:X^3:$ in (1.1):

$$:X^3: = Y_\infty^3 + 3Y_\infty^2 Z_\infty + 3Y_\infty :Z_\infty^2: + :Z_\infty^3:.$$

We also have

$$:X^3: = Y^3 + 3Y^2 Z + 3Y :Z^2: + :Z^3:$$

if we set

$$:Z^3: = -(e^{At}Z_\infty(0))^3 + 3(e^{At}Z_\infty(0))^2 Z_\infty - 3(e^{At}Z_\infty(0)) :Z_\infty^2: + :Z_\infty^3:.$$

We will see that Y and Y_∞ has positive regularity whereas Z has the same smoothness as Z_∞. Using the product rules above, it can be seen that all products above make sense.

Finally, we define $Z_N = \Pi_N Z$ and introduce the following approximation of (1.1):

$$\begin{cases} \dfrac{dY_N}{dt} = AY_N - (Y_N^3 + 3Y_N^2 Z_N + 3Y_N {:}\, Z_N^2 {:} + {:}\, Z_N^3 {:}), \\ \\ Y_N(0) = x. \end{cases} \tag{2.2}$$

Equivalently, setting $X_N = Y_N + Z_N$, we may write with a slight abuse of notation

$$: X_N^3 : = Y_N^3 + 3Y_N^2 Z_N + 3Y_N : Z_N^2 : + : Z_N^3 :$$

and

$$\begin{cases} dX_N = (AX_N + {:}\, X_N^3 {:})dt + \Pi_N dW, \\ \\ X_N(0) = x. \end{cases} \tag{2.3}$$

Note that this is not a Galerkin approximation. However, since $: X_N^3 : = X_N^3 - 3\rho_N^2 X_N$ and $\Pi_N dW$ is a finite dimensional white noise, it is classical to prove that X_N exists and is unique. Moreover, it is smooth in space when the initial datum is smooth; in this case Y_N is also spatially smooth and has moreover regularity $3/2^-$ in time. All the computations done in the sections below are rigorous when the initial datum is smooth, for initial data in a negative Besov space, they are obtained thanks to a preliminary smoothing of the initial datum.

It is not difficult to prove that (X_N) converges to the unique solution of (1.1) (see [TsWe18]) in convenient topologies, $C([0,T]; C^{-\alpha})$ for $\alpha > 0$ for instance.

Let us define the Gibbs measure ν on $\mathcal{H}$

$$\nu(dx) = \mathcal{Z}^{-1} e^{-2U(x)} \mu(dx), \tag{2.4}$$

where

$$U(x) = \frac{1}{4} \int_H : x^4 : (\xi) d\xi \tag{2.5}$$

and

$$\mathcal{Z} := \int_H e^{-2U(x)} \mu(dx).$$

Then

$$e^{-2U(x)} \in L^r(\mathcal{H}, \mu),$$

for any $r \in [1, \infty)$ thanks to the Nelson inequality.

We can write equation (1.1) as

$$\begin{cases} \dfrac{dY}{dt} = AY - Y^3 - 3Y^2 Z - 3Y : Z^2 : - : Z^3 :, \\ \\ Y(0) = x, \end{cases} \tag{2.6}$$

or, in mild form,

$$Y(t) = e^{At}x - \int_0^t e^{A(t-s)} [Y^3(s) + 3Y^2(s)Z(s) + 3Y(s) : Z^2(s) : + : Z^3(s) :] ds. \tag{2.7}$$

We define the transition semigroups P_t and P_t^N associated to (1.1) and (2.3) respectively and introduce the following modified semigroup:

$$S_t^N \varphi(x) = \mathbb{E}\left(e^{-K \int_0^t V(X_N(s)) ds} \varphi(X_N(t,x)) \right) \tag{2.8}$$

where the potential V is given by

$$V(X_N) = \left| : X_N^2 : \right|_{H^{-\alpha}}^p = \left| X_N^2 - \rho_N^2 \right|_{H^{-\alpha}}^p$$

for some $K > 0$, $p > 0$, $\alpha > 0$ to be chosen. This quantity is well defined for $\varphi \in \mathcal{B}_b(C^{-\alpha})$, the space of borelian bounded functions on $C^{-\alpha}$.

The following relation formally follows from Duhamel formula since $P_N\varphi$ satisfies the Kolmogorov equation associated to (2.3) and $S_N\varphi$ satisfies the same equation with an extra term due to the potential V:

$$P_t^N\varphi(x) = S_t^N\varphi(x) + K\int_0^t S_{t-s}^N(VP_s\varphi)(x)ds. \tag{2.9}$$

It can be proved rigorously by an approximation argument.

3. Estimates

Let us consider the equation:

$$\begin{cases} \dfrac{d\eta^{h,N}}{dt} = (A\eta^{h,N} - 3 : X_N^2 : \eta^{h,N}), \\ \eta^{h,N}(0) = h. \end{cases} \tag{3.1}$$

It is classical that $\eta^{h,N} = D_x X_N \cdot h$ is the differential of X_N with respect to the intial data $x \in C^{-\alpha}$ in the direction $h \in C^{-\alpha}$.

In this section, we derive several estimates on η_N and X_N. These hold only on finite time intervals and we restrict to $t \in [0,1]$.

Lemma 3.1. *Let $\alpha < \frac{1}{3}$, $\epsilon > 0$ such that $\alpha + \epsilon < \frac{1}{3}$ and $p \geq 1$, $s > 0$ such that $\alpha + \epsilon + \frac{2}{3p} + \frac{s}{3} < \frac{1}{3}$ then there exists c depending on α, ϵ, p such that for all $t \in (0,1]$, $x \in C^{-\alpha}$, $h \in C^{-s}$*

$$|\eta^{h,N}(t)|_{C^{\alpha+\epsilon}} \leq ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}}\, e^{c\int_0^t |:X_N^2(s):|_{H^{-\alpha}}^p\, ds}$$

and

$$|\eta^{h,N}(t)|_{L^\infty} \leq ct^{-s/2}|h|_{C^{-s}}\, e^{c\int_0^t |:X_N^2(s):|_{H^{-\alpha}}^p\, ds}.$$

Proof. Write (3.1) in mild form

$$\eta^{h,N}(t) = e^{At}h - 3\int_0^t e^{A(t-s)}\left(: X_N^2(s) : \eta^{h,N}(s)\right) ds.$$

Below we omit the dependence on N. By the smoothing of the heat semigroup e^{At}, the product rule in Besov spaces and $t \in [0,1]$, we have

$$|\eta^h(t)|_{C^{\alpha+\epsilon}} \leq ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left|: X^2(r) : \eta^h(r)\right|_{H^{-\alpha-\epsilon/2}} dr$$

$$\leq ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left|: X^2(r) :\right|_{H^{-\alpha}} \left|\eta^h(r)\right|_{C^{\alpha+\epsilon}} dr.$$

Let $\lambda(t) = t^{(\alpha+\epsilon+s)/2} \left|\eta^h(t)\right|_{C^{\alpha+\epsilon}}$, then since $t \in [0,1]$,

$$\lambda(t) \leq c|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\, r^{-(\alpha+\epsilon+s)/2} \left|: X^2(r) :\right|_{H^{-\alpha}} \lambda(r)dr.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\frac{(3\alpha + 3\epsilon + 1 + s)q}{2} < 1, \quad i.e. \ \frac{3\alpha + 3\epsilon + 1 + s}{2} < 1 - \frac{1}{p}.$$

Then

$$\begin{aligned}\lambda^p(t) &\leq c|h|^p_{C^{-s}} + c\left(\int_0^t |t-r|^{-q(\alpha+\epsilon+1/2)}\, r^{-q(\alpha+\epsilon+s)/2}\, dr\right)^{p/q} \int_0^t \left|: X^2(r) :\right|^p_{H^{-\alpha}} \lambda^p(r) dr \\ &\leq c|h|^p_{C^{-s}} + c\int_0^t \left|: X^2(r) :\right|^p_{H^{-\alpha}} \lambda^p(r) dr\end{aligned}$$

The first inequality follows by Gronwall's lemma.

For the second inequality, we write:

$$|\eta^h(t)|_{L^\infty} \leq ct^{-s/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left|: X^2(r) :\, \eta^h(r)\right|_{H^{-\alpha-\epsilon/2}} dr$$

$$\leq ct^{-s/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left|: X^2(r) :\right|_{H^{-\alpha}} \left|\eta^h(r)\right|_{C^{\alpha+\epsilon}} dr.$$

and with p, q as above:

$$\begin{aligned}|\eta^h(t)|^p_{L^\infty} &\leq ct^{-sp/2}|h|^p_{C^{-s}} + c\int_0^t r^{p(\alpha+\epsilon+s)/2} \left|: X^2(r) :\right|^p_{H^{-\alpha}} \left|\eta^h(r)\right|^p_{C^{\alpha+\epsilon}} dr \\ &\leq ct^{-sp/2}|h|^p_{C^{-s}} + c\int_0^t \left|: X^2(r) :\right|^p_{H^{-\alpha}} e^{cp\int_0^t r\left|:X^2_N(s):\right|^p_{H^{-\alpha}} ds} |h|^p_{C^{-s}} dr \\ &\leq c|h|^p_{C^{-s}} \left(t^{-sp/2} + e^{cp\int_0^t \left|:X^2_N(s):\right|^p_{H^{-\alpha}} ds}\right).\end{aligned}$$

The conclusion follows. □

We now estimate moments of Y_N solution of (2.2). We estimate Y_N in $L^r(\mathbb{T}^2)$, $r \geq 2$, using similar arguments as in [TsWe18] and [MoWe15], but in a simplified way since we do not need such refined estimates for large times.

Lemma 3.2. *Let $\alpha \in [0,1)$ and $r \geq 2$, there exist $c > 0$ depending on α and r such that*

$$\frac{1}{r}\frac{d}{dt}|Y_N|^r_{L^r} + \frac{r-1}{r^2}\left|\nabla((Y_N)^{r/2})\right|^2_{L^2} + \frac{1}{2}|Y_N|^{r+2}_{L^{r+2}} \leq c\,\mathcal{L}_\alpha(Z_N)^{k_r},$$

where $k_r = \dfrac{r+2}{1-\alpha}$ and

$$\mathcal{L}_\alpha(Z_N) = |Z_N|_{C^{-\alpha}} + |:Z_N^2:|_{C^{-\alpha}} + |:Z_N^3:|_{C^{-\alpha}} + 1.$$

Proof. Again we omit the dependence on N. Multiplying equation (2.2) by Y^{r-1} and integrating on $\mathbb{T}^2$ we obtain:

$$\frac{1}{r}\frac{d}{dt}|Y|^r_{L^r} + \frac{4(r-1)}{r^2}\left|\nabla(Y^{r/2})\right|^2_{L^2} + |Y|^{r+2}_{L^{r+2}} = -\int_{\mathbb{T}^2}(3Y^{r+1}\,Z + 3Y^r\,:Z^2: + Y^{r-1}\,:Z^3:)d\xi.$$

We estimate each of the three terms in the right hand side. By Proposition A.8 and Proposition A.9 in [TsWe18], we have

$$3\left|\int_{\mathbb{T}^2} Y^{r+1}\,Z\,d\xi\right| \leq 3|Y^{r+1}|_{B^\alpha_{1,1}}|Z|_{C^{-\alpha}} \leq c\left(|Y^{r+1}|^{1-\alpha}_{L^1}|\nabla Y^{r+1}|^\alpha_{L^1} + |Y^{r+1}|_{L^1}\right)|Z|_{C^{-\alpha}}.$$

Clearly

$$|Y^{r+1}|_{L^1} = |Y|^{r+1}_{L^{r+1}} \leq c|Y|^{r+1}_{L^{r+2}}.$$

Moreover

$$|\nabla Y^{r+1}|_{L^1} = (r+1)|Y^r \nabla Y|_{L^1} \leq (r+1)|Y^{\frac{r+2}{2}}|_{L^2}\, |Y^{\frac{r-2}{2}}\nabla Y|_{L^2} = \frac{2(r+1)}{r}|Y|^{\frac{r+2}{2}}_{L^{r+2}}\, |\nabla Y^{r/2}|_{L^2}.$$

It follows that

$$3\left|\int_{\mathbb{T}^2} Y^{r+1}\, Z\, d\xi\right| \leq c\left(|Y|^{(1-\alpha)(r+1)+\alpha(r+2)/2}_{L^{r+2}}\, |\nabla Y^{r/2}|^{\alpha}_{L^2} + |Y|^{r+1}_{L^{r+2}}\right)\, |Z|_{C^{-\alpha}}$$

$$= c\left(|Y|^{((1-\alpha/2)r+1}_{L^{r+2}}\, |\nabla Y^{r/2}|^{\alpha}_{L^2} + |Y|^{r+1}_{L^{r+2}}\right)\, |Z|_{C^{-\alpha}}$$

$$\leq \frac{1}{6}\,|Y|^{r+2}_{L^{r+2}} + \frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2} + c\left(|Z|^{p_1}_{C^{-\alpha}} + |Z|^{p_2}_{C^{-\alpha}}\right)$$

with

$$\frac{(1-\alpha/2)r+1}{r+2} + \frac{\alpha}{2} + \frac{1}{p_1} = 1$$

and $p_2 = r+2$. Note that since $\alpha \in (0,1)$, p_1 exists and is equal to $\dfrac{r+2}{1-\alpha}$. We proceed similarly for the second one and obtain:

$$3\left|\int_{\mathbb{T}^2} Y^r\, :Z^2:\, d\xi\right| \leq \frac{1}{6}\,|Y|^{r+2}_{L^{r+2}} + \frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2} + c\left(|:Z^2:|^{p_3}_{C^{-\alpha}} + |:Z^2:|^{p_4}_{C^{-\alpha}}\right)$$

with

$$\frac{\alpha}{2} + \frac{r(1-\alpha/2)}{r+2} + \frac{1}{p_3} = 1$$

and $p_4 = \frac{r+2}{2}$ and for the third one

$$\left|\int_{\mathbb{T}^2} Y^{r-1}\, :Z^3:\, d\xi\right| \leq \frac{1}{6}\,|Y|^{r+2}_{L^{r+2}} + \frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2} + c\left(|:Z^3:|^{p_5}_{C^{-\alpha}} + |:Z^3:|^{p_6}_{C^{-\alpha}}\right)$$

with

$$\frac{\alpha}{2} + \frac{r(1-\alpha/2)-1}{r+2} + \frac{1}{p_5} = 1$$

and $p_6 = \frac{r+2}{3}$.

Using these three inequalities in the energy equality and noting that $p_1 = \frac{r+2}{1-\alpha} \geq p_i$, $i = 2, ..., 5$ we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,|Y|^r_{L^r} + \frac{r-1}{r^2}\left|\nabla(Y^{r/2})\right|^2_{L^2} + \frac{1}{2}\,|Y|^{r+2}_{L^{r+2}} \leq c\left(|Z|_{C^{-\alpha}} + |:Z^2:|_{C^{-\alpha}} + |:Z^3:|_{C^{-\alpha}} + 1\right)^{p_1}.$$

This gives the result with $k_r = p_1$. □

Lemma 3.3. *Let $\alpha < \frac{1}{3}$, $k \geq 1$ there exists $C_{\alpha,k} > 0$ and $\kappa_{\alpha,k} > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,1]} |Y_N(t)|^k_{C^{-\alpha}}\right) \leq C_{\alpha,k}\left(|x|_{C^{-\alpha}} + 1\right)^{\kappa_{\alpha,k}}.$$

Proof. Since

$$|Y_N|^{r+2}_{L^r} \leq c|Y_N|^{r+2}_{L^{r+2}}$$

it follows from Lemma 3.2 that

$$\frac{d}{dt}\,|Y_N(t)|^r_{L^r} + c|Y_N(t)|^{r+2}_{L^r} \leq c\,\mathcal{L}_\alpha(Z_N(t))^{k_r}$$

and by [TsWe18, Lemma 3.8] it follows for $t \in [0,1]$

$$|Y_N(t)|_{L^r} \leq c \max\left(t^{-r/2}, \sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}}\right).$$

We choose $r > \frac{2}{\alpha}$ and use the embedding $L^r \subset C^{-\alpha}$ to deduce

$$|Y_N(t)|_{C^{-\alpha}} \leq c \max\left(t^{-r/2}, \sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}}\right). \tag{3.2}$$

It remain to obtain a bound for $|Y_N(t)|_{C^{-\alpha}}$ for small time. Write the mild form for the equation satisfied by Y_N:

$$Y_N(t) = e^{At}x - \int_0^t e^{A(t-s)}(Y_N^3(s) + 3Y_N^2(s)Z_N(s) + 3Y_N(s) :Z_N^2(s): + :Z_N^3(s):)ds$$

Let $\alpha < \frac{1}{3}$, $\beta > \alpha$ such that

$$\gamma := \frac{\alpha+\beta}{2} < \frac{1}{3}$$

and define

$$M(t) = \sup_{s\in[0,t]} (|Y_N(s)|_{C^{-\alpha}} + s^\gamma |Y_N(s)|_{C^{-\beta}}).$$

Then similar computations as in the proof of Theorem 3.3 in [TsWe18] give for $t \in [0,1]$:

$$M(t) \leq c_1|x|_{C^{-\alpha}} + c_2 t^{1-3\gamma} M(t)^3 + c_3 t^{1-3\gamma} \sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t))^3,$$

for some constants c_1, c_2 which can be chosen larger than 1.

Therefore if we set

$$t^* = \min\left\{\left(c_3 \sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t))^3\right)^{-\frac{1}{1-3\gamma}}, \left(8c_2(c_1|x|_{C^{-\alpha}}+1)^2\right)^{-\frac{1}{1-3\gamma}}, 1\right\}$$

we have

$$\sup_{t\in[0,t^*]} M(t) \leq 2(c_1|x|_{C^{-\alpha}} + 1) \tag{3.3}$$

It follows from (3.2) and (3.3):

$$\begin{aligned}\sup_{t\in[0,1]} |Y_N(t)|_{C^{-\alpha}} &\leq \sup_{t\in[0,t^*]} |Y_N(t)|_{C^{-\alpha}} + \sup_{t\in[t^*,1]} |Y_N(t)|_{C^{-\alpha}} \\ &\leq 2(c_1|x|_{C^{-\alpha}}+1) + \max\left((t^*)^{-2/r}, \sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{rk_r}{r+2}}\right)\end{aligned}$$

The result follows taking the expectation and using finiteness of the moments of

$$\sup_{t\in[0,1]} \mathcal{L}_\alpha(Z_N(t)).$$

□

We set

$$\widetilde{Z}_N(t) = e^{At}x + \Pi_N \int_0^t e^{A(t-s)}\, dW(s) = e^{At}x + Z_N(t),$$

and

$$\widetilde{Y}_N(t) = X_N(t) - \widetilde{Z}_N(t).$$

Then $\widetilde{Y}_N$ satisfies the same equation as Y_N, see (2.2), with Z_N replaced by $\widetilde{Z}_N$ and 0 initial datum.

Lemma 3.4. *Let $\alpha \in (0,1/9)$, $k \geq 1$ then*

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\widetilde{Y}|_{L^2}^2 + \int_0^t |\nabla \widetilde{Y}_N(s)|_{L^2}^2\, ds\right)^k \leq C_{\alpha,k}\left(|x|_{C^{-\alpha}}^{12k/(1-\alpha)} + 1\right)$$

for any $t > 0$.

Proof. Lemma 3.2 clearly also holds for $\widetilde{Y}_N$ provided Z_N is replaced by $\widetilde{Z}_N$. We use it with $r = 2$ and again omit to indicate the dependency on N:

$$\frac{1}{2}\frac{d}{dt}|\widetilde{Y}|_{L^2}^2 + \frac{1}{4}\left|\nabla\widetilde{Y}\right|_{L^2}^2 + \frac{1}{2}\left|\widetilde{Y}\right|_{L^4}^4 \leq c\,\mathcal{L}_\alpha(\widetilde{Z})^{k_2},$$

c an $k_2 = \frac{4}{1-\alpha}$ are given in Lemma 3.2. We use the product rules in Besov spaces and the smoothing effect of e^{At} to prove:

$$\mathcal{L}_\alpha(\widetilde{Z}(s)) \leq C_{\alpha,\epsilon}\left((s^{-(2\alpha+\epsilon)}+1)\,|x|_{C^{-\alpha}}^3 + 1\right)\mathcal{L}_\alpha(Z(s))$$

the result follows by integrating on $[0,T]$ and taking expectation of the k^{th} power. Note that since $\alpha < 1/9$, it is possible to choose $\epsilon > 0$ small enough $(2\alpha+\epsilon)k_2 < 1$.

□

Lemma 3.5. *Let $\alpha \in (0,1/9)$, $k \geq 1$, there exists $\gamma_{\alpha,k}, C_{\alpha,k}$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\widetilde{Y}_N(t)|_{H^{\alpha+\epsilon}}^k\right) \leq C_{\alpha,k}\left(|x|_{C^{-\alpha}}+1\right)^{\gamma_{\alpha,k}}.$$

Proof. We write, omitting again N

$$\widetilde{Y}(t) = \int_0^t e^{A(t-s)}\left(\widetilde{Y}^3(s) + 3\widetilde{Y}^2(s)\widetilde{Z}(s) + 3\widetilde{Y}(s)\,:\widetilde{Z}^2(s):\, +\, :\widetilde{Z}^3(s):\right)ds$$

and get

$$|\widetilde{Y}(t)|_{H^{\alpha+\epsilon}} \leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\left(|\widetilde{Y}^3|_{H^{-\alpha}} + |\widetilde{Y}^2\widetilde{Z}|_{H^{-\alpha}} + |\widetilde{Y}\,:\widetilde{Z}^2:|_{H^{-\alpha}} + |:\widetilde{Z}^3:|_{H^{-\alpha}}\right)ds.$$

We have

$$\begin{aligned}
|\widetilde{Y}^3|_{H^{-\alpha}} &\leq c|\widetilde{Y}^2|_{H^{\alpha+\epsilon}}\,|\widetilde{Y}|_{C^{-\alpha}} \leq c|\widetilde{Y}|_{B^{\alpha+\epsilon}_{4,4}}^2\,|\widetilde{Y}|_{C^{-\alpha}} \leq c|\widetilde{Y}|_{H^{\alpha+\epsilon+1/2}}^2\,|\widetilde{Y}|_{C^{-\alpha}} \\
&\leq c\left(|\widetilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)}|\nabla\widetilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\widetilde{Y}|_{L^2}^2\right)|\widetilde{Y}|_{C^{-\alpha}}.
\end{aligned}$$

Similarly

$$|\widetilde{Y}^2\widetilde{Z}|_{H^{-\alpha}} \leq c|\widetilde{Y}^2|_{H^{\alpha+\epsilon}}\,|\widetilde{Z}|_{C^{-\alpha}} \leq c\left(|\widetilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)}|\nabla\widetilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\widetilde{Y}|_{L^2}^2\right)|\widetilde{Z}|_{C^{-\alpha}}$$

and

$$|\widetilde{Y}\,:\widetilde{Z}^2:|_{H^{-\alpha}} \leq c|\widetilde{Y}|_{H^{\alpha+\epsilon}}\,|:\widetilde{Z}^2:|_{C^{-\alpha}} \leq c\left(|\widetilde{Y}|_{L^2}^{1-(\alpha+\epsilon)}|\nabla\widetilde{Y}|_{L^2}^{\alpha+\epsilon} + |\widetilde{Y}|_{L^2}\right)|:\widetilde{Z}^2:|_{C^{-\alpha}}.$$

Gathering these results gives

$$\begin{aligned}
|\widetilde{Y}(t)|_{H^{\alpha+\epsilon}} \leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\Big[&\left(|\widetilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)}|\nabla\widetilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\widetilde{Y}|_{L^2}^2\right)\left(|\widetilde{Y}|_{C^{-\alpha}} + |\widetilde{Z}|_{C^{-\alpha}}\right) \\
&+\left(|\widetilde{Y}|_{L^2}^{1-(\alpha+\epsilon)}|\nabla\widetilde{Y}|_{L^2}^{\alpha+\epsilon} + |\widetilde{Y}|_{L^2}\right)\,|:\widetilde{Z}^2:|_{C^{-\alpha}} + |:\widetilde{Z}^3:|_{C^{-\alpha}}\Big]ds =: (a)+(b)+(c).
\end{aligned}$$

The first term is bounded thanks to Hölder inequality

$$(a) = c\int_0^t (t-s)^{-(\alpha+\epsilon/2)} \Big[\left(|\widetilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)} |\nabla\widetilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\widetilde{Y}|_{L^2}^2 \right) (|\widetilde{Y}|_{C^{-\alpha}} + |\widetilde{Z}|_{C^{-\alpha}}) ds$$

$$\leq c \left(|\widetilde{Y}|_{L^{p_1}(0,t,L^2)}^{2(1/2-\alpha-\epsilon)} (|\nabla\widetilde{Y}|_{L^2(0,t,L^2)}^{2(1/2+\alpha+\epsilon)} + |\widetilde{Y}|_{L^{p_2}(0,t,L^2)}) \right) \left(|Y|_{L^{p_3}(0,t,C^{-\alpha})} + |x|_{C^{-\alpha}} + |Z|_{L^{p_3}(0,t,C^{-\alpha})} \right)$$

with

$$\frac{2(1/2-\alpha-\epsilon)}{p_1} + (1/2+\alpha+\epsilon) + \alpha + \frac{\epsilon}{2} < 1 + \frac{1}{p_3}$$

and

$$\frac{2}{p_2} + \alpha + \frac{\epsilon}{2} + \frac{1}{p_3} < 1.$$

Since $\alpha < \frac{1}{4}$, ϵ, p_1, p_2, p_3 can be chosen such this is satisfied. Then

$$\begin{aligned}
(b) \quad &\leq \quad c\int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(|\widetilde{Y}|_{L^2}^{1-(\alpha+\epsilon)} |\nabla\widetilde{Y}|_{L^2}^{\alpha+\epsilon} + |\widetilde{Y}|_{L^2} \right) \\
&\times \quad \left[s^{-(\alpha+\epsilon/2)} \left(|x|_{C^{-\alpha}}^2 + |x|_{C^{-\alpha}} |Z|_{C^{-\alpha}} \right) + |:Z^2:|_{C^{-\alpha}} \right] ds \\
&\leq \quad c \left(|\widetilde{Y}|_{L^{p_4}(0,t,L^2)}^{1-(\alpha+\epsilon)} |\nabla\widetilde{Y}|_{L^2(0,t,L^2)}^{\alpha+\epsilon} + |\widetilde{Y}|_{L^{p_5}(0,t,L^2)} \right) \\
&\times \quad \left(|x|_{C^{-\alpha}}^2 + |x|_{C^{-\alpha}} |Z|_{L^{p_6}(0,t,C^{-\alpha})} + |:Z^2:|_{L^{p_6}(0,t,C^{-\alpha})} \right)
\end{aligned}$$

with

$$\alpha + \frac{\epsilon}{2} + \frac{1-(\alpha+\epsilon)}{p_4} + \frac{\alpha+\epsilon}{2} + \alpha + \frac{\epsilon}{2} + \frac{1}{p_6} < 1$$

and

$$2(\alpha + \frac{\epsilon}{2}) + \frac{1}{p_5} + \frac{1}{p_6} < 1$$

which again is possible since $\alpha < \frac{1}{4}$. Similarly

$$(c) \leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(s^{-(2\alpha+\epsilon)} (|x|_{C^{-\alpha}}^3 + |x|_{C^{-\alpha}}^2 |Z|_{C^{-\alpha}}) \right.$$

$$\left. + s^{-(\alpha+\epsilon/2)} |x|_{C^{-\alpha}} |:Z^2:|_{C^{-\alpha}} + |:Z^3:|_{C^{-\alpha}} \right) ds$$

Taking expectation we find thanks to Lemma 3.3, Lemma 3.4, Hölder inequality and finiteness of the moments of $\sup_{t\in[0,1]} |Z|_{C^{-\alpha}}$, $\sup_{t\in[0,1]} |:Z^2:|_{C^{-\alpha}}$ and $\sup_{t\in[0,1]} |:Z^3:|_{C^{-\alpha}}$:

$$\mathbb{E}\left(\sup_{t\in[0,1]} |\widetilde{Y}(t)|_{H^{\alpha+\epsilon}}^k \right) \leq c \left(|x|_{C^{-\alpha}}^{\kappa_{\alpha,k}} + 1 \right)$$

for some $\kappa_{\alpha,k}$ which could be written explicitly. □

Lemma 3.6. *Let $\alpha \in (0, 1/9)$, for any $k \geq 1$, there exists $\overline{\gamma}_{\alpha,k}, C_{\alpha,k}$ such that*

$$\mathbb{E}\left(|:X_N(t)^2:|_{H^{-\alpha}}^k ds \right) \leq C_{\alpha,k}(t^{-\alpha k} + 1) \left(|x|_{C^{-\alpha}} + 1 \right)^{\overline{\gamma}_{\alpha,k}}.$$

Proof. We have, omitting the dependence in N,

$$\begin{aligned}
|:X(t)^2:|_{H^{-\alpha}} &\leq c\left(|Y^2(t)|_{H^{-\alpha}} + |Y(t)|_{H^{\alpha+\epsilon}}|Z(t)|_{C^{-\alpha}} + |:Z(t)^2:|_{C^{-\alpha}}\right)\\
&\leq c\left(|Y(t)|_{H^{\alpha+\epsilon}}\left(|Y(t)|_{C^{-\alpha}} + |Z(t)|_{C^{-\alpha}}\right) + |:Z(t)^2:|_{C^{-\alpha}}\right)\\
&\leq c\left(\left(t^{-\alpha}|x|_{C^{-\alpha}} + |\widetilde{Y}(t)|_{H^{\alpha+\epsilon}}\right)\left(|Y(t)|_{C^{-\alpha}} + |Z(t)|_{C^{-\alpha}}\right) + |:Z(t)^2:|_{C^{-\alpha}}\right)
\end{aligned}$$

The result follows by Lemma 3.3 and Lemma 3.5. □

3.1. **Estimates on S_t^N.** We have

$$S_t^N\varphi(x) = \mathbb{E}\left(e^{-K\int_0^t V(X_N(s))ds}\varphi(X_N(t,x))\right)$$

where the potential V is given by

$$V(x) = \left|:x^2:\right|^p_{H^{-\alpha}}.$$

Lemma 3.7. *Let $\alpha \in (0, \frac{1}{9})$, $s > 0$, $p \geq 1$ such that $\alpha + \frac{1}{3p} + \frac{s}{3} < \frac{1}{3}$ and $\alpha(2p-1) + s < 2$. Then there exists γ such that for any $\varphi \in \mathcal{B}_b(C^{-\alpha})$ we have for $t \in [0,1]$, $x \in C^{-\alpha}$, $h \in H$*

$$|DS_t^N\varphi(x)\cdot h| \leq c(t^{-(1+s+2\alpha p)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma$$

and

$$|DS_t^N(V\varphi)(x)\cdot h| \leq c(t^{-(1+s+2\alpha p)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma.$$

Remark 3.8. *$V\varphi$ is not bounded but it is clear from Lemma 3.6 that S_t^N can be extended to borelian functions with polynomial growth.*

Proof. We only prove the second inequality, the first one is similar.

We omit to mention the dependence of N. From [DaZa97] we know that $S_t^N\varphi$ is differentiable in any direction $h \in H$ and we have the following modified Bismut–Elworthy–Li formula:

$$\begin{aligned}
DS_t(V\varphi)(x)\cdot h &= \frac{1}{t}\,\mathbb{E}\left(e^{-K\int_0^t V(X(s,x))ds}V(X(t,x))\varphi(X(t,x))\int_0^t(\eta^h(s),dW(s))\right)\\
&\quad +2K\mathbb{E}\left(e^{-K\int_0^t V(X(s,x))ds}V(X(t,x))\varphi(X(t,x))\int_0^t(1-\tfrac{s}{t})\,V'(X(s,x))\cdot\eta^h(s)\,ds\right)\\
&= (a)+(b).
\end{aligned}$$

Clearly

$$\begin{aligned}
(a) &\leq \frac{1}{t}\,\|\varphi\|_0\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2}\\
&\quad\times\left[\mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t(\eta^h(s),dW(s))\right|^2\right)\right]^{1/2}.
\end{aligned}$$

By Lemma 3.6 we have for K large enough

$$\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2} \leq c(t^{-\alpha p}+1)(1+|x|_{C^{-\alpha}})^{\overline{\gamma}_{\alpha,2p}/2}.$$

To estimate the second factor, we proceed as in the proof of [DaDe03-2, Lemma 4.1]. Writing Itô's formula for

$$d\left|e^{-K\int_0^t V(X(s,x)ds}\int_0^t(\eta^h(s),dW(s))\right|^2$$

we deduce for K large enough

$$\mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t(\eta^h(s),dW(s))\right|^2\right)^{1/2}$$

$$\leq \mathbb{E}\left(\int_0^t e^{-2K\int_0^s V(X(r,x)dr}\left|\eta^h(s)\right|^2 ds\right)^{1/2}$$

$$\leq c\mathbb{E}\left(\int_0^t \sigma^{-s}\,ds\right)^{1/2}|h|_{C^{-s}}$$

thanks to Lemma 3.1 and the embedding $L^\infty \subset L^2$. In Lemma 3.1, we choose $\epsilon > 0$ such that $\alpha+\epsilon+\frac{1}{3p}+\frac{s}{3}<\frac{1}{3}$.

It follows that

$$(a) \leq c\,(t^{-(1+s)/2}+1)(t^{-\alpha p}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma_{\alpha,2p}/2}.$$

Similarly, the other term is bounded by

$$(b) \quad \leq c\|\varphi\|_0\,\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2}\,\mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t V'(X(s,x))\cdot\eta^h(s)\,ds\right|^2\right)^{1/2}$$

We have

$$\begin{aligned}|V'(X(s,x))\cdot\eta^h(s)| &= 2p|(:X^2(s):|^{p-2}_{H^{-\alpha}}\,|(:X^2(s):,X(s)\eta^h(s))|_{H^{-\alpha}}\\ &\leq 2p|:X^2(s):|^{p-1}_{H^{-\alpha}}\,|X(s)\eta^h(s)|_{H^{-\alpha}}\\ &\leq 2p|:X^2(s):|^{p-1}_{H^{-\alpha}}\,|X(s)|_{H^{-\alpha}}\,|\eta^h(s)|_{C^{\alpha+\epsilon}}.\end{aligned}$$

By Lemma 3.1, Minkowski inequality, Lemma 3.3 and Lemma 3.5, we thus can write for K large enough:

$$\begin{aligned}&E\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t V'(X(s,x))\cdot\eta^h(s)\,ds\right|^2\right)^{1/2}\\ &\leq c\mathbb{E}\left(\left(\int_0^t|:X^2(s):|^{p-1}_{H^{-\alpha}}\,|X(s)|_{H^{-\alpha}}s^{-(\alpha+\epsilon+s)/2}\,ds\right)^2\right)^{1/2}|h|_{C^{-s}}\\ &\leq c\int_0^t\mathbb{E}\left(\left(|:X^2(s):|^{p-1}_{H^{-\alpha}}\,|X(s)|_{H^{-\alpha}}\right)^2\right)^{1/2}s^{-(\alpha+\epsilon+s)/2}\,ds|h|_{C^{-s}}\\ &\leq c\left(\int_0^t(1+s^{-(\alpha+\epsilon+s+2\alpha(p-1))/2})\,ds\right)(1+|x|_{C^{-\alpha}})^{(\overline{\gamma}_{\alpha,2p}/2+\overline{\gamma}_{\alpha,4(p-1)}+\kappa_{\alpha,4})/4}|h|_{C^{-s}}.\end{aligned}$$

We deduce for $\epsilon>0$ small enough so that $\alpha+\epsilon+s+2\alpha(p-1))<2$:

$$(b)\leq c\|\varphi\|_0(1+t^{-\alpha p})(1+|x|_{C^{-\alpha}})^{(\overline{\gamma}_{\alpha,4(p-1)}+\kappa_{\alpha,4})/4}|h|_{C^{-s}}$$

The conclusion follows when gathering the estimates on (a) and (b).

□

3.2. **Estimate on P_t^N.** Recall that

$$P_t^N\varphi(x) = S_t^N\varphi(x) + K\int_0^t S_{t-s}^N(VP_s^N\varphi)ds.$$

Theorem 3.9. *Let $s \in (0,1)$ and $\alpha \in (0,1/9)$ such that $\frac{2\alpha}{1-s} < 1-s-3\alpha$. Let $p \geq 1$ such that $\frac{2\alpha}{1-s} < \frac{1}{p} < 1-s-3\alpha$, then there exists $\gamma > 0$, and $c > 0$ depending on s, α and p such that for any $\varphi \in \mathcal{B}_b(C^{-\alpha})$ we have for $t \in [0,1]$, $x \in C^{-\alpha}$, $h \in H$:*

$$|DP_t^N\varphi(x)\cdot h| \leq c(t^{-(1+s+2\alpha p)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma.$$

Remark 3.10. *It is easy to find α, s satisfying the assumptions of Theorem 3.9. For instance, choose any $s \in (0,1)$, $\delta \in (0, \frac{1-s}{2})$ and $\alpha < \min\{\frac{\delta(1-s)}{1+3\delta}; s\}$. Then*

$$\frac{\alpha}{\delta} < 1-s-3\alpha.$$

Moreover

$$\frac{\alpha}{\delta} > \frac{2\alpha}{1-s}.$$

It follows that one may choose p such that $\frac{2\alpha}{1-s} < \frac{\alpha}{\delta} < \frac{1}{p} < 1-s-3\alpha$ and Theorem 3.9 implies

$$|DP_t^N\varphi(x)\cdot h| \leq c(t^{-(1+s+\delta)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma.$$

Remark 3.11. *Since H is dense in C^{-s}, this result shows that $DP_t^N\varphi(x)$ belongs to the dual of C^{-s} and the bound extends to $h \in C^{-s}$.*

Proof. Since for $t \geq 1$, $\|P_t\varphi\|_0 \leq \|\varphi\|_0$, it suffices to prove the result for $t \in [0,1]$ and use the Markov property.

Since $\frac{2\alpha}{1-s} > \frac{2\alpha}{2-s+\alpha}$, we have $\frac{2\alpha}{2-s+\alpha} < \frac{1}{p} < 1-s-3\alpha$ and this is equivalent to the assumptions of Lemma 3.7, therefore we may write:

$$\begin{aligned}
|DP_t^N\varphi(x)\cdot h| &\leq |DS_t^N\varphi(x)\cdot h| + K\int_0^t |DS_{t-s}^N(VP_s\varphi)(x)|ds \\
&\leq c(t^{-(1+s+2\alpha p)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma \\
&\quad + K\int_0^t c\,(t-s)^{-(1+s+2\alpha p)/2}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma\,ds.
\end{aligned}$$

This implies the result since $1+s+2\alpha p < 2$. □

Letting $N \to \infty$ in Theorem 3.9, we deduce:

Corollary 3.12. *Let $s \in (0,1)$ and $\alpha \in (0,1/9)$ such that $\frac{2\alpha}{1-s} < 1-s-3\alpha$. Let $p \geq 1$ such that $\frac{2\alpha}{1-s} < \frac{1}{p} < 1-s-3\alpha$, then there exists $\gamma > 0$, and $c > 0$ depending on s, α and p such that for any $\varphi \in \mathcal{B}_b(H)$ we have for $t \in [0,1]$, x, $y \in C^{-\alpha}$:*

$$|P_t\varphi(x) - P_t\varphi(y)| \leq c(t^{-(1+s+2\alpha p)/2}+1)\,|x-y|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}}+|y|_{C^{-\alpha}})^\gamma.$$

4. The Kolmogorov operator

Given $N \in \mathbb{N}$, we denote by $\mathcal{F}_N C_b^\infty$ the set of all functions $\varphi : H \to \mathbb{R}$ of the form

$$\varphi(x) = g(x_j,\ |j| \leq N), \tag{4.1}$$

where $g : \mathbb{R}^{(2N+1)^2} \to \mathbb{R}$, $(\xi_j,\, |j| \le N) \to g(\xi_j,\, |j| \le N)$ is of class C_b^∞. Moreover, we set

$$\mathcal{F}C_b^\infty = \bigcap_{N=1}^{\infty} \mathcal{F}_N C_b^\infty.$$

Note that if $\varphi \in \mathcal{F}_N C_b^\infty$ is of the form (4.1) we have

$$D\varphi(x) = \sum_{|k|\le N} D_k g(\xi_j,\, |j| \le N) e_k \tag{4.2}$$

and

$$D^2\varphi(x) = \sum_{|k|\le N, |h|\le N} D_h D_k g(\xi_j,\, |j| \le N)\, e_h \otimes e_k. \tag{4.3}$$

Let us introduce the Kolmogorov operator on $\mathcal{F}C_b^\infty$, setting

$$\mathcal{K}\varphi := \frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi] + \langle Ax - : x^3 :, D\varphi\rangle, \quad \varphi \in \mathcal{F}C_b^\infty. \tag{4.4}$$

This definition is meaningful because if φ is given by (4.2) we have

$$\langle : x^3 :, D\varphi\rangle = \sum_{|k|\le N} \langle D_k g(\xi_j,\, |j| \le N) e_k, : x^3 :\rangle.$$

We set

$$\rho(x) = Z^{-1} e^{-2U(x)},$$

so that

$$D\log\rho(x) = -2DU(x) = -2 : x^3 : . \tag{4.5}$$

We also consider the approximate measure ν_N in H

$$\nu_N(dx) = Z_N^{-1} e^{-U_N(x)} \mu_N(dx), \tag{4.6}$$

where

$$U_N(x) = \frac{1}{4}\int_H : x_N^4 : (\xi) d\xi \tag{4.7}$$

and

$$Z_N := \int_H e^{-U_N(x)} \mu_N(dx).$$

Note that

$$\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\varphi + |D\varphi|^2, \quad \forall\, \varphi \in \mathcal{F}_N C_b^\infty. \tag{4.8}$$

Integrating with respect to ν over $\mathcal{H}$, yields

$$\int_{\mathcal{H}} \mathcal{K}\varphi\, \varphi\, d\nu = -\frac{1}{2}\int_{\mathcal{H}} |D\varphi|^2 d\nu, \quad \forall\, \varphi \in \mathcal{F}C_b^\infty. \tag{4.9}$$

We also introduce an approximating operator $\mathcal{K}_N$ on $\mathcal{F}C_b^\infty$ setting, for $\varphi(x) = g(x_j,\, |j| \le M)$

$$\begin{aligned}
\mathcal{K}_N\varphi &:= \frac{1}{2}\,\mathrm{Tr}\,[\Pi_N\, D^2\varphi] + \langle Ax - : x_N^3 :, D\varphi\rangle \\
&= \frac{1}{2}\sum_{|h|\le N\wedge M} D_h^2 g(\xi_j,\, |j| \le M) - \sum_{|h|\le M} (\alpha_h - \rho_N) x_h D_h g(\xi_j,\, |j| \le M) \\
&- \sum_{|h|\le M}\ \sum_{|j_1|,|j_2|,|j_3|\le M} x_{j_1} x_{j_2} x_{j_3} \delta_{h,j_1+j_2+j_3} D_h g(\xi_j,\, |j| \le M),
\end{aligned} \tag{4.10}$$

where $\alpha_h = (1 + |h|^2)$.

Similarly as above, we have

$$\int_{\mathcal{H}} \mathcal{K}_N\varphi\,\varphi\,d\nu_N = -\frac{1}{2}\int_{\mathcal{H}} |D\varphi|^2 d\nu_N, \quad \forall\,\varphi \in \mathcal{F}_N C_b^\infty. \tag{4.11}$$

4.1. m–**dissipativity of** $\mathcal{K}$**.** Fix $N_0 \in \mathbb{N}$ and let $f \in \mathcal{F}_{N_0} C_b^\infty$. For $N \in \mathbb{N}$ we consider the solution φ_N of the approximating equation

$$\lambda\varphi_N - \frac{1}{2}\,\mathrm{Tr}\,[D^2\Pi_N\varphi_N] + \langle Ax_N - : x_N^3 :, D\varphi_N\rangle = f. \tag{4.12}$$

Then, taking into account (4.11) we find the estimate

$$\int_H |D\varphi_N|^2\,d\nu_N \le \frac{2}{\lambda}\int_H f^2\,d\nu_N. \tag{4.13}$$

Theorem 4.1. *$\mathcal{K}$ is extendible to an m–dissipative operator in $L^1(\mathcal{H},\nu)$ whose extension we denote by $\mathcal{K}^{(1)}$. Moreover, $\mathcal{F}C_b^\infty$ is a core for $\mathcal{K}^{(1)}$.*

Proof. We have only to show

$$\lim_{N\to\infty}\int_{\mathcal{H}} \langle :x_N^3: - :x^3:, D\varphi_N\rangle d\nu = 0 \tag{4.14}$$

and then to apply the Lumer–Phillips theorem in some Besov spaces.

We take α, s and p satisfying the assumptions of Theorem 3.9 and write

$$|DP_t^N f(x)\cdot h| \le c\,t^{-(1+s+\alpha p)/2}+1)\,|h|_{C^{-s}}\|f\|_0\,(1+|x|_{C^{-\alpha}})^\gamma, \quad t \ge 0,$$

for some $\alpha < \frac{1}{8}$, $\epsilon > 0$ and c, γ depending on s, α, ϵ. Moreover, we can choose $1+\alpha+\epsilon+s < 2$.

We write

$$\varphi_N(x) = \int_0^\infty e^{-\lambda t} P_t^N f(x) ds$$

and deduce

$$|D\varphi_N(x)\cdot h| \le c|h|_{C^{-s}}\,\|f\|_0(1+|x|_{C^{-\alpha}})^\gamma.$$

It then suffices to write

$$\left|\int_{\mathcal{H}} \langle :x_N^3: - :x^3:\rangle D\varphi_N\right| d\nu \le c\int_{\mathcal{H}} |:x_N^3: - :x^3:|_{C^{-s}}\,\|f\|_0(1+|x|_{C^{-\alpha}})^\gamma\,d\nu$$

By the Hölder inequality this converges to 0.

□

Remark 4.2. Arguing as in [DaDe07] one can show that the closure $\mathcal{K}^{(p)}$ of $\mathcal{K}$ in $L^p(\mathcal{H},\nu)$, $p \ge 1$ is m–dissipative. Moreover the gradient operator

$$D : \mathcal{F}C_b^1 \subset L^p(\mathcal{H},\nu) \to L^p(\mathcal{H},\nu;H)$$

is closable. We denote by $\overline{D}_p$ its closure and by $W^{1,p}(\mathcal{H},\nu)$ its domain.

Finally,

$$D(\mathcal{K}^{(p)}) \subset W^{1,2}(\mathcal{H},\nu)$$

and it results

$$\int_{\mathcal{H}} \mathcal{K}^{(2)}\varphi\,\varphi\,d\nu = -\frac{1}{2}\int_{\mathcal{H}} |D\varphi|^2 d\nu, \quad \forall\,\varphi \in D(\mathcal{K}^{(2)}). \tag{4.15}$$

Moreover since, as easily checked

$$\int_{\mathcal{H}} \mathcal{K}^{(2)}\varphi\,\psi\,d\nu = -\frac{1}{2}\int_{\mathcal{H}} \langle D\varphi, D\psi\rangle d\nu, \quad \forall\,\varphi,\,\psi \in \mathcal{F}\,C_b^2(H),$$

it results

$$\int_{\mathcal{H}} \mathcal{K}^{(2)}\varphi\,\psi\,d\nu = -\frac{1}{2}\int_{\mathcal{H}} \langle D\varphi, D\psi\rangle d\nu, \quad \forall\ \varphi,\ \psi \in D(\mathcal{K}^{(2)}), \tag{4.16}$$

so that $\mathcal{K}^{(2)}$ is *symmetric*. Finally, let us recall the classical integration by parts formula for the Gaussian measure μ

$$\int_{\mathcal{H}} \langle D\varphi, z\rangle\,\psi\,d\mu = -\int_{\mathcal{H}} \langle D\psi, z\rangle\,\varphi\,d\mu + \int_{\mathcal{H}} W_z\,\psi\,\varphi\,d\mu, \quad h \in \mathbb{Z}^2,\ \psi,\ \varphi\ \in \mathcal{F}C_b^1(\mathcal{H}), \tag{4.17}$$

where

$$W_z(x) = \langle Q^{-1/2}x, z\rangle = \sum_{k\in\mathbb{Z}^2} \sqrt{1+|k|^2}\ z_k x_k, \quad \phi \in \mathcal{H}$$

and Q is the covariance of μ. Since $\nu(dx) = \rho(x)\mu(dx)$, setting $\psi = \rho$ in (4.17) yields

$$\int_{\mathcal{H}} \langle D\varphi, z\rangle\,\rho\,d\mu = -\int_{\mathcal{H}} \langle D\rho, z\rangle\,\varphi\,d\mu + \int_{\mathcal{H}} W_z\,\rho\,\varphi\,d\mu, \quad h \in \mathbb{Z}^2,\ \varphi\ \in \mathcal{F}C_b^1(\mathcal{H}).$$

which is equivalent to

$$\int_{\mathcal{H}} \langle D\varphi, z\rangle\,d\nu = -\int_{\mathcal{H}} \langle D\log\rho, z\rangle\,\varphi\,d\nu + \int_{\mathcal{H}} W_z\,\varphi\,d\nu, \quad h \in \mathbb{Z}^2,\ \varphi\ \in \mathcal{F}C_b^1(\mathcal{H}).$$

Since $\langle D\log\rho, z\rangle = -\langle :x^3:, z\rangle$ we have

$$\int_{\mathcal{H}} \langle D\varphi, z\rangle\,d\nu = \int_{\mathcal{H}} \langle :x^3:, z\rangle\,\varphi\,d\nu + \int_{\mathcal{H}} W_z\,\varphi\,d\nu, \quad h \in \mathbb{Z}^2,\ \varphi\ \in \mathcal{F}C_b^1(\mathcal{H}).$$

Therefore, for any $p > 1$ and any $z \in \mathbb{Z}^2$ there is $C_p > 0$ such that

$$\left|\int_{\mathcal{H}} \langle D\varphi, z\rangle\,d\nu\right| \le C_p\|\varphi\|_{L^p(\mathcal{H},\nu)}\,|z|. \tag{4.18}$$

This inequality is useful to develop a geometrical theory for the measure ν, see [DaDe17], [DaDe16] and [DaLuTu18].

References

[BaChDa11] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, vol. 343 of Grundlehren der Mathematischen Wissenschaften. Springer, Heidelberg, 2011.

[DaDe03-1] G. Da Prato and A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 877–947.

[DaDe03-2] G. Da Prato and A. Debussche, *Strong solutions to the stochastic quantization equations*, *Ann. Probab.,* **31**, 2003, no. 4, 1900–1916.

[DaDe07] G. Da Prato and A. Debussche, *m–Dissipativity of Kolmogorov Operators Corresponding to Burgers Equations with Space-time White Noise*, Potential Anal. **26** 31–35, 2007.

[DaDe16] G. Da Prato and A. Debussche, *Estimate for $P_t D$ for the stochastic Burgers equation*, Ann. Inst. H. Poincaré Probab. Stat. **52**, 12481258, 2016.

[DaDe17] G. Da Prato and A. Debussche, *An integral inequality for the invariant measure of a stochastic reaction–diffusion equation*, J. Evol. Equ. **17**, 197-214, 2017.

[DaLuTu18] G. Da Prato, A. Lunardi and L. Tubaro, *Malliavin Calculus for non gaussian differentiable measures and surface measures in Hilbert spaces*, Trans. Amer. Math. Soc. 370, no. 8, 5795–5842, 2018.

[DaZa97] G. Da Prato and J. Zabczyk, *Differentiability of the Feynman–Kac semigroup and a control application*, Rend. Mat. Accad. Lincei 8, 183–188, 1997.

[Ha14] M. Hairer, *A theory of regularity structures.* Invent. Math. 198 (2) (2014) 269–504.

[HaMa16] M. Hairer and J. Mattingly. *The strong Feller property for singular stochastic PDEs.*, Ann. Inst. Henri Poincarè Probab. Stat. **54**, no. 3, 1314–1340, 2018.

[MoWe15] J. C. Mourrat and H. Weber, *Global well-posedness of the dynamic Φ^4 in the plane.* Ann. Probab. **45**), no. 4, 2398–2476, 2017.

[TsWe18] P. Tsatsoulis and H. Weber, *Spectral gap for the stochastic quantization equation on the 2-dimensional torus*, Annales de l'Institut Henri Poincaré - Probabilités et Statistiques 2018, Vol. 54, No. 3, 1204–1249

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