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## ON GENUS ONE MIRROR SYMMETRY IN HIGHER DIMENSIONS AND THE BCOV CONJECTURES

#### DENNIS ERIKSSON, GERARD FREIXAS I MONTPLET, AND CHRISTOPHE MOUROUGANE

ABSTRACT. The mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (BCOV) proposed, in a seminal article from '94, a conjecture extending genus zero mirror symmetry to higher genera. With a view towards a refined formulation of the Grothendieck–Riemann–Roch theorem, we offer a mathematical description of the BCOV conjecture at genus one. As an application of the arithmetic Riemann–Roch theorem of Gillet–Soulé and of our previous results on the BCOV invariant, we establish this conjecture for Calabi–Yau hypersurfaces in projective spaces. Our contribution takes place on the B-side, and together with the work of Zinger on the A-side, it provides the first complete examples of the mirror symmetry program in higher dimensions. The case of quintic threefolds was studied by Fang–Lu–Yoshikawa. Our approach also lends itself to arithmetic considerations of the BCOV invariant, and we study a Chowla–Selberg type theorem expressing it in terms of special  $\Gamma$  values for certain Calabi–Yau manifolds with complex multiplication.

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#### 1. Introduction

The purpose of this article is to establish higher dimensional cases of genus one mirror symmetry, as envisioned by mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (henceforth abbreviated BCOV) in their influential paper [BCOV94]. Precisely, we relate the generating series of genus one Gromov–Witten invariants on Calabi–Yau hypersurfaces to an invariant of a mirror family, built out of holomorphic analytic torsions. The invariant, whose existence was conjectured in *loc. cit.*, was mathematically defined and studied in our previous paper [EFiMM18a]. We refer to it as the BCOV invariant  $\tau_{BCOV}$ . In dimension 3, the construction of the BCOV invariant and its relation to mirror symmetry were established by Fang–Lu–Yoshikawa [FLY08], relying on previous results by [Zin08, Zin09].

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Our approach parallels the Kodaira–Spencer formulation of the Yukawa coupling in genus zero, and can be recast as a refined version of the Grothendieck–Riemann–Roch theorem à la Deligne [Del87]. We hope this point of view will also be inspiring to study higher genus Gromov–Witten invariants and the *B*-side of mirror symmetry in dimension 3. In this setting, the *A*-side has received a lot of attention recently.

1.1. The classical BCOV conjecture at genus one. Let X be a Calabi–Yau manifold of dimension n. In this article, this will mean a complex projective connected manifold with trivial canonical sheaf. We now briefly recall the BCOV program at genus one.

On the one hand, on what is referred to as the A-side, we consider enumerative invariants associated to X. For this, recall first that for every curve class  $\beta$  in  $H_2(X,\mathbb{Z})$ , there is a proper Deligne–Mumford stack of stable maps from genus g curves to X, whose fundamental class is  $\beta$ :

$$\overline{\mathcal{M}}_g(X,\beta) = \{f: C \to X \mid g(C) = g, f \text{ stable and } f_*[C] = \beta\}.$$

The virtual dimension of this stack can be computed to be (cf. [Beh97], in particular the introduction)

$$\int_{\beta} c_1(X) + (\dim(X) - 3)(1 - g) = (\dim(X) - 3)(1 - g).$$

Whenever dim(X) = 3 or g = 1 this is of virtual dimension 0 and one can consider the Gromov–Witten invariants

$$GW_g(X, \beta) = \deg \left[ \overline{\mathcal{M}}_g(X, \beta) \right]^{\text{vir}} \in \mathbb{Q}.$$

Since the main focus of our paper is higher dimension, we henceforth impose g = 1. One then defines the formal power series

(1.1) 
$$F_1^A(\tau) = \frac{-1}{24} \int_X c_{n-1}(X) \cap 2\pi i \tau + \sum_{\beta > 0} GW_1(X, \beta) e^{2\pi i \langle \tau, \beta \rangle},$$

where  $\tau$  belongs to the complexified Kähler cone  $\mathcal{H}_X$ . <sup>1</sup>

On the other hand, on what is referred to as the B-side, BCOV introduced a spectral quantity  $\mathscr{F}_1^B$  built out of holomorphic Ray–Singer analytic torsions of a mirror Calabi–Yau manifold  $X^{\vee}$ . It depends on an auxiliary choice of a Kähler structure  $\omega$  on  $X^{\vee}$ , and can be recast as

$$\mathscr{F}_1^B(X^{\vee},\omega) = \prod_{0 \le p,q \le n} (\det \Delta^{p,q}_{\overline{\partial}})^{(-1)^{p+q}pq},$$

where  $\det \Delta^{p,q}_{\overline{\partial}}$  is the  $\zeta$ -regularized determinant of the Dolbeault Laplacian acting on  $A^{p,q}(X^{\vee})$ . In our previous work [EFiMM18a] we normalized this quantity to make it independent of the choice of  $\omega$ :

$$\tau_{BCOV}(X^{\vee}) = C(X^{\vee}, \omega) \cdot \mathcal{F}_1^B(X^{\vee}, \omega),$$

for some explicit the constant  $C(X^{\vee}, \omega)$ . Thus  $\tau_{BCOV}(X^{\vee})$  only depends on the complex structure of the Calabi–Yau manifold, in accordance with the philosophy that the B-model only depends on variations of the complex structure on  $X^{\vee}$ .

Mirror symmetry predicts that given X, there is a mirror family of Calabi–Yau manifolds over a punctured multi-disc around the origin  $\varphi \colon \mathscr{X}^{\vee} \to \mathbf{D}^{\times} = (\mathbb{D}^{\times})^d$ , with maximally unipotent monodromies and  $d = h^{1,1}(X) = h^1(T_{X^{\vee}})$ . Here we denoted by  $X^{\vee}$  any member of the mirror family.

 $<sup>{}^1</sup>$ If  $\mathcal{K}_X$  denotes the Kähler cone of X, we define  $\mathcal{H}_X$  as  $H^{1,1}_{\mathbb{R}}(X)/H^{1,1}_{\mathbb{Z}}(X)+i\mathcal{K}_X$ .

<sup>&</sup>lt;sup>2</sup>Such families are also called *large complex structure limits* of Calabi–Yau manifolds.

The A-side and the B-side should be related by a distinguished biholomorphism onto its image  $\mathbf{D}^{\times} \to \mathcal{H}_X$ , which is referred to as the mirror map and is denoted  $q \mapsto \tau(q)$ . The mirror map sends the origin of the multi-disc to infinity. Fixing a basis of ample classes on X, we can think of it as a change of coordinates on  $\mathbf{D}^{\times}$ . In the special case of d = 1, one such a map is constructed as a quotient of carefully selected periods in [Mor93].

**BCOV** conjecture at genus one. Let X be a Calabi–Yau manifold and  $\varphi \colon \mathscr{X}^{\vee} \to \mathbf{D}^{\times}$  a mirror family as above. Then:

- (1) there is a procedure, called passing to the holomorphic limit, to extract from  $\tau_{BCOV}(\mathscr{X}_q^{\vee})$  as  $q \to 0$  a holomorphic function  $F_1^B(q)$ .
- (2) the functions  $F_1^A$  and  $F_1^B$  are related via the mirror map by

$$F_1^B(q)=F_1^A(\tau(q)).$$

Passing to the holomorphic limit is often interpreted as considering a Taylor expansion of  $\tau_{BCOV}(\mathcal{X}_q^{\vee})$  in  $\tau(q)$  and  $\overline{\tau(q)}$ , and keeping the holomorphic part. In this article, we will instead use a procedure based on degenerations of Hodge structures.

1.2. **Grothendieck–Riemann–Roch formulation of the BCOV conjecture at genus one.** The purpose of this subsection is to formulate a version of the BCOV conjecture producing the holomorphic function  $F_1^B$  without any reference to spectral theory, holomorphic anomaly equations or holomorphic limits. Our formulation parallels the Hodge theoretic approach to the Yukawa coupling in 3-dimensional genus zero mirror symmetry: the key ingredients going into its construction are the Kodaira–Spencer mappings between Hodge bundles, and canonical trivializations of those (cf. [Mor93]).

To state our conjecture, we need to introduce the BCOV line bundle  $\lambda_{BCOV}(\mathscr{X}^{\vee}/\mathbf{D}^{\times})$  of the mirror family  $\varphi \colon \mathscr{X}^{\vee} \to \mathbf{D}^{\times}$ . The BCOV line of a Calabi–Yau manifold  $X^{\vee}$  is defined to be

$$\lambda_{BCOV}(X^{\vee}) = \bigotimes_{0 \le p, q \le n} \det H^{q}(X^{\vee}, \Omega_{X^{\vee}}^{p})^{(-1)^{p+q}p}.$$

For a family of Calabi–Yau manifolds it glues together to a holomorphic line bundle on the base. Also, we denote by  $\chi$  the Euler characteristic of any fiber of  $\varphi$  and by  $K_{\mathcal{X}^{\vee}/\mathbf{D}^{\times}}$  the relative canonical bundle.

**Refined BCOV conjecture at genus one.** Let X be a Calabi-Yau manifold and  $\varphi \colon \mathscr{X}^{\vee} \to \mathbf{D}^{\times}$  a mirror family as in §1.1. Then:

(1) there exists a natural isomorphism of line bundles,

(1.2) 
$$\mathbf{GRR} \colon \lambda_{BCOV}(\mathscr{X}^{\vee}/\mathbf{D}^{\times})^{\otimes 12\kappa} \xrightarrow{\sim} \varphi_{*}(K_{\mathscr{X}^{\vee}/\mathbf{D}^{\times}})^{\otimes \chi\kappa},$$

together with natural trivializing sections of both sides. Here  $\kappa$  is a non-zero integer which only depends on the relative dimension of  $\varphi$ .

(2) The isomorphism **GRR** can be realized as a holomorphic function, which when written as  $\exp((-1)^n F_1^B(q))^{24\kappa}$  satisfies

$$F_1^B(q) = F_1^A(\tau(q)).$$

The existence of some isomorphism in (1.2) is provided by the Grothendieck–Riemann–Roch theorem in Chow theory, the key point of the conjecture being the naturality requirement. In fact, an influential program by Deligne [Del87] suggests that the codimension one part of the

usual Grothendieck–Riemann–Roch equality can be lifted to a base change invariant isometry of line bundles, when equipped with natural metrics. An intermediate version of this exists via the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS92], which provides an equality of isometry classes of hermitian line bundles. Properly interpreted, this establishes a link between the BCOV invariant and a metric evaluation of (1.2).

A more detailed treatment of the formulation of the conjecture is given in Section 6. Examples related to the existing literature are also discussed.

1.3. **Main results.** In this subsection we discuss the framework and statements of our results. For Calabi–Yau hypersurfaces in projective space, our main theorem settles the BCOV conjecture and its refinement.

Let X be a Calabi–Yau hypersurface in  $\mathbb{P}^n_{\mathbb{C}}$ , with  $n \geq 4$ . Its complexified Kähler cone is onedimensional, induced by restriction from that of the ambient projective space. The mirror family  $f: \mathcal{Z} \to U$  can be realized using a crepant resolution of the quotient of the Dwork pencil

(1.3) 
$$x_0^{n+1} + \dots + x_n^{n+1} - (n+1)\psi x_0 \dots x_n = 0, \quad \psi \in U = \mathbb{C} \setminus \mu_{n+1},$$

by the subgroup of  $\mathsf{GL}_{n+1}(\mathbb{C})$  given by  $G = \{g \cdot (x_0, \ldots, x_n) = (\xi_0 x_0, \ldots, \xi_n x_n), \xi_i^{n+1} = 1, \prod \xi_i = 1\}$ . Moreover,  $f \colon \mathcal{Z} \to U$  can be naturally extended across  $\mu_{n+1}$  to a degeneration with ordinary double point singularities, sometimes referred to as a conifold degeneration.

The monodromy around  $\psi=\infty$  is maximally unipotent and the properties of the limiting Hodge structure can be used to define a natural flag of homology cycles. Using this we can produce natural holomorphic trivializations  $\widetilde{\eta}_k$ , in a neighborhood of  $\psi=\infty$ , of the primitive Hodge bundles  $(R^k f_* \Omega_{\mathcal{Z}/U}^{n-1-k})_{\text{prim}}$ , which have unipotent lower triangular period matrices. These sections have natural  $L^2$  norms given by Hodge theory. The product  $\bigotimes_{k=0}^{n-1} \widetilde{\eta}_k^{(n-1-k)(-1)^{n-1}}$  is the essential building block of a natural frame  $\widetilde{\eta}_{BCOV}$  of  $\lambda_{BCOV}(\mathcal{Z}/U)$ .

Finally, let  $F_1^A(\tau(\psi))$  be the generating series defined as in (1.1), for a general Calabi–Yau hypersurface  $X \subset \mathbb{P}^n_{\mathbb{C}}$ . Here  $\psi \mapsto \tau(\psi)$  is the mirror map. Then our main result (Theorem 5.9 and Theorem 6.3) can be stated as follows: <sup>3</sup>

**Main Theorem.** Let  $n \ge 4$ . Consider a Calabi–Yau hypersurface  $X \subset \mathbb{P}^n_{\mathbb{C}}$  and the mirror family  $f: \mathcal{Z} \to U$  above. Then:

(1) in a neighborhood of infinity, the BCOV invariant of  $Z_{V}$  factors as

$$\tau_{BCOV}(Z_{\psi}) = C \left| \exp\left( (-1)^{n-1} F_1^B(\psi) \right) \right|^4 \left( \frac{\|\widetilde{\eta}_0\|_{L^2}^{\chi(Z_{\psi})/12}}{\|\widetilde{\eta}_{BCOV}\|_{L^2}} \right)^2,$$

where  $F_1^B(\psi)$  is a multivalued holomorphic function with  $F_1^B(\psi) = F_1^A(\tau(\psi))$  as formal series in  $\psi$ , and C is a positive constant.

(2) up to a constant, the refined BCOV conjecture at genus one is true for X and its mirror family, with the choices of trivializing sections  $\tilde{\eta}_{BCOV}$  and  $\tilde{\eta}_0$ .

Actually, the theorem also holds in the case of cubic curves (as follows from \$1.5) and quartic surfaces. We also show, more generally, in Proposition 6.4 that the refined BCOV conjecture holds, up to a constant, for K3 surfaces.

 $<sup>\</sup>overline{{}^3}$ To facilitate the comparison with the BCOV conjecture, notice that *X* has now dimension n-1 instead of *n*.

The first part of the theorem extends to arbitrary dimensions previous work of Fang–Lu–Yoshikawa [FLY08, Thm. 1.3] in dimension 3. In their approach, all the Hodge bundles have geometric meaning in terms of Weil–Petersson geometry and Kuranishi families. The lack thereof is an additional complication in our setting.

To our knowledge, our theorem is the first complete example of higher dimensional mirror symmetry, of BCOV type at genus one, established in the mathematics literature. It confirms various instances that had informally been utilized for computational purposes, *e.g.* [KP08, Sec. 6] in dimension 4.

#### 1.4. Overview of proof of the main theorem.

Arithmetic Riemann–Roch. In the algebro-geometric setting, the arithmetic Riemann–Roch theorem from Arakelov theory allows us to compute the BCOV invariant of a family of Calabi–Yau varieties in terms of  $L^2$  norms of auxiliary sections of Hodge bundles. This bypasses some arguments in former approaches, such as [FLY08], based on the holomorphic anomaly equation (cf. [EFiMM18a, Proposition 5.9]). It determines the BCOV invariant up to a meromorphic function, in fact a rational function. The divisor of this rational function is encapsulated in the asymptotics of the  $L^2$  norms and the BCOV invariant. In the special case when the base is a Zariski open set of  $\mathbb{P}^1_{\mathbb{C}}$ , as for the Dwork pencil (1.3) and the mirror family, this divisor is determined by all but one point. Hence so is the function itself, up to constant. The arithmetic Riemann–Roch theorem simultaneously allows us to establish the existence of an isomorphism **GRR** as in (1.2).

Hodge bundles of the mirror family. The construction of the auxiliary sections is first of all based on a comparison of the Hodge bundles of the mirror family with the G-invariant part of the Hodge bundles on the Dwork pencil (1.3), explained in Section 3. Using the residue method of Griffiths we construct algebraic sections of the latter. These are then transported into sections  $\eta_k$  of the Hodge bundles of the crepant resolution, *i.e.* the mirror family. This leads us to a systematic geometric study of these sections in connection with Deligne extensions and limiting Hodge structures at various key points, notably at  $\mu_{n+1}$  where ordinary double point singularities arise. We rely heavily on knowledge of the Yukawa coupling and our previous work in [EFiMM18a, Sec. 2] on logarithmic Hodge bundles and semi-stable reduction. The arguments are elaborated in Section 4.

Asymptotics of  $L^2$  norms and the BCOV invariant. The above arithmetic Riemann-Roch reduction leads us to study the norm of the auxiliary sections outside of the maximally unipotent monodromy point, enabling us to focus on ordinary double points. Applying our previous result [EFiMM18a, Thm. 4.4] to the auxiliary sections, we find that the behaviour of their  $L^2$  norms is expressed in terms of monodromy eigenvalues, and the possible zeros or poles as determined by the geometric considerations of the preceding paragraph. The monodromy is characterized by the Picard–Lefschetz theorem. As for the asymptotics for the BCOV invariant, they were already accomplished in [EFiMM18a, Thm. B]. This endeavor results in Theorem 5.1, which is a description of the rational function occurring in the arithmetic Riemann–Roch theorem.

<sup>&</sup>lt;sup>4</sup>This rational function compares to the so-called *holomorphic ambiguity* in the physics literature.

Connection to enumerative geometry. The BCOV conjecture suggests that we need to study the BCOV invariant close to  $\psi = \infty$ . However, the formula in Theorem 5.1 is not adapted to the mirror symmetry setting, for example the sections  $\eta_k$  do not make any reference to  $H_{\text{lim}}^{n-1}$ . We proceed to normalize the  $\eta_k$  by dividing by holomorphic periods, for a fixed basis of the weight filtration on the homology  $(H_{n-1})_{\text{lim}}$ , to obtain the sections  $\tilde{\eta}_k$  of the main theorem. Rephrasing Theorem 5.1 with these sections, we thus arrive at an expression for the  $F_1^B$  in the theorem. Combined with results of Zinger [Zin08, Zin09], this yields the relation to the generating series of Gromov–Witten invariants in the mirror coordinate. Lastly, the refined BCOV conjecture is deduced in this case through a reinterpretation of the BCOV invariant and the arithmetic Riemann–Roch theorem.

#### 1.5. Applications to Kronecker limit formulas.

Classical first Kronecker limit formula. The simplest Calabi–Yau varieties are elliptic curves, which can conveniently be presented as  $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ , for  $\tau$  in the Poincaré upper half-plane. The generating series (1.1) of Gromov–Witten invariants is then given by  $-\frac{1}{24}\log\Delta(\tau)$ , where  $\Delta(\tau)=q\prod(1-q^n)^{24}$  and  $q=e^{2\pi i\tau}$ . The corresponding function  $\mathscr{F}_1^B$  is computed as  $\exp(\zeta_\tau'(0))$ , where

$$\zeta_{\tau}(s) = (2\pi)^{-2s} \sum_{(m,n)\neq(0,0)} \frac{(\operatorname{Im} \tau)^s}{|m+n\tau|^{2s}}.$$

The BCOV conjecture at genus one is deduced from the equality

(1.4) 
$$\exp(-\zeta_{\tau}'(0)) = \frac{1}{(2\pi)^2} \operatorname{Im}(\tau) |\Delta(\tau)|^{1/6}.$$

This is a formulation of the first Kronecker limit formula, see e.g. [Yos99, Intro.]. In the mirror symmetry interpretation, the correspondence  $\tau \mapsto q$  is the (inverse) mirror map. Equation (1.4) can be recovered from a standard application of the arithmetic Riemann–Roch theorem. In this vein, we will interpret all results of this shape as generalizations of the Kronecker limit formula. This includes the Theorem 5.1 cited above, as well as a Theorem 2.6 for Calabi–Yau hypersurfaces in Fano manifolds.

Chowla–Selberg formula. While being applicable to algebraic varieties over  $\mathbb{C}$ , the Riemann–Roch theorem in Arakelov geometry has the further advantage of providing arithmetic information when the varieties are defined over  $\mathbb{Q}$ .

The arithmetic Riemann–Roch theorem is suited to evaluating the BCOV invariant of certain arithmetically defined Calabi–Yau varieties with additional automorphisms. As an example, for the special fibre  $Z_0$  of our mirror family (1.3), Theorem 7.2 computes the BCOV invariant as a product of special values of the  $\Gamma$  function. This is reminiscent of the Chowla–Selberg theorem [SC67], which derives from (1.4) an expression of the periods of a CM elliptic curve as a product of special  $\Gamma$  values. Assuming deep conjectures of Gross–Deligne [Gro78], we would be able to write any BCOV invariant of a CM Calabi–Yau manifold in such terms.

#### 2. THE BCOV INVARIANT AND THE ARITHMETIC RIEMANN-ROCH THEOREM

2.1. **Kähler manifolds and**  $L^2$  **norms.** Let X be a compact complex manifold. In this article, a hermitian metric on X means a smooth hermitian metric on the holomorphic vector bundle

 $T_X$ . Let h be a hermitian metric on X. The Arakelov theoretic Kähler form attached to h is given in local holomorphic coordinates by

(2.1) 
$$\omega = \frac{i}{2\pi} \sum_{j,k} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) dz_j \wedge d\overline{z}_k.$$

We assume that the complex hermitian manifold (X, h) is Kähler, that is the differential form  $\omega$  is closed.

The hermitian metric h induces hermitian metrics on the  $\mathscr{C}^{\infty}$  vector bundles of differential forms of type (p,q), that we still denote h. Then, the spaces  $A^{p,q}(X)$  of global sections, inherit a  $L^2$  hermitian inner product

(2.2) 
$$h_{L^2}(\alpha, \beta) = \int_X h(\alpha, \beta) \frac{\omega^n}{n!}.$$

The coherent cohomology groups  $H^q(X,\Omega_X^p)$  can be computed as Dolbeault cohomology, that in turn can be computed in  $A^{p,q}(X)$  by taking  $\overline{\partial}$ -harmonic representatives. Via this identification,  $H^q(X,\Omega_X^p)$  inherits a  $L^2$  inner product. Similarly, the hermitian metric h also induces hermitian metrics on the vector bundles and spaces of complex differential forms of degree k. The complex de Rham cohomology  $H^k(X,\mathbb{C})$  has an induced  $L^2$  inner product by taking d-harmonic representatives. The canonical Hodge decomposition

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p,q}^{\perp} H^q(X,\Omega_X^p)$$

is an isometry for the  $L^2$  metrics.

2.2. **The BCOV invariant.** We briefly recall the construction of the BCOV invariant [EFiMM18b, Sec. 5]. Let X be a Calabi–Yau manifold of dimension n. Fix a Kähler metric h on X, with Kähler form  $\omega$  as in (2.1). Let  $T(\Omega_X^p, \omega)$  be the holomorphic analytic torsion of the vector bundle  $\Omega_X^p$  of holomorphic differential p-forms endowed with the metric induced by h, and with respect to the Kähler form  $\omega$  on X. The BCOV torsion of  $(X, \omega)$  is

$$T(X,\omega) = \prod_{0 \le p \le n} T(\Omega_X^p, \omega)^{(-1)^p p}.$$

Let  $\Delta^{p,q}_{\overline{\partial}}$  be the Dolbeault Laplacian acting on  $A^{p,q}(X)$ , and  $\det \Delta^{p,q}_{\overline{\partial}}$  its  $\zeta$ -regularized determinant (excluding the zero eigenvalue). Unraveling the definition of holomorphic analytic torsion, we find for the BCOV torsion

$$T(X,\omega) = \prod_{0 \le p,q \le n} (\det \Delta \frac{p,q}{\overline{\partial}})^{(-1)^{p+q}pq}.$$

It depends on the choice of the Kähler metric. A suitable normalization makes it independent of choices. For this purpose, we introduce two real valued quantities. For the first one, let  $\eta$  be a basis of  $H^0(X, K_X)$ , and define as in [FLY08, Sec. 4]

(2.3) 
$$A(X,\omega) = \exp\left(-\frac{1}{12} \int_X (\log \varphi) c_n(T_X, h)\right), \quad \text{with} \quad \varphi = \frac{i^{n^2} \eta \wedge \overline{\eta}}{\|\eta\|_{L^2}^2} \frac{n!}{(2\pi\omega)^n}.$$

For the second one, we consider the largest torsion free quotient of the cohomology groups  $H^k(X,\mathbb{Z})$ , denoted by  $H^k(X,\mathbb{Z})_{\mathrm{nt}}$ . These are lattices in the real cohomology groups  $H^k(X,\mathbb{R})$ . The latter have Euclidean structures induced from the  $L^2$  inner products on the  $H^k(X,\mathbb{C})$ . We

define  $\operatorname{vol}_{L^2}(H^k(X,\mathbb{Z}),\omega)$  to be the square of the covolume of the lattice  $H^k(X,\mathbb{Z})_{\mathrm{nt}}$  with respect to this Euclidean structure, and we put

(2.4) 
$$B(X,\omega) = \prod_{0 \le k \le 2n} \operatorname{vol}_{L^2}(H^k(X,\mathbb{Z}),\omega)^{(-1)^{k+1}k/2}.$$

The *BCOV invariant* of *X* is then defined to be

(2.5) 
$$\tau_{BCOV}(X) = \frac{A(X,\omega)}{B(X,\omega)} T(X,\omega) \in \mathbb{R}_{>0}.$$

The BCOV invariant depends only on the complex structure of X [EFiMM18a, Prop. 5.8]. The definition (2.5) differs from that of [EFiMM18a, Def. 5.7] by a factor  $(2\pi)^{n^2\chi(X)/2}$ , due to the different choice of normalization of the  $L^2$  metric:

$$\langle \alpha, \beta \rangle = \int_X h(\alpha, \beta) \frac{(2\pi\omega)^n}{n!}.$$

2.3. **The arithmetic Riemann–Roch theorem.** In this section, we work over an arithmetic ring. This means an excellent regular domain A together with a finite set  $\Sigma$  of embeddings  $\sigma: A \hookrightarrow \mathbb{C}$ , closed under complex conjugation. For example, A could be a number field with the set of all its complex embeddings, or the complex field  $\mathbb{C}$ . Denote by K the field of fractions of A.

Let X be an arithmetic variety, i.e. a regular, integral, flat and quasi-projective scheme over A. For every embedding  $\sigma \colon A \hookrightarrow \mathbb{C}$ , the base change  $X_{\sigma} = X \times_{A,\sigma} \mathbb{C}$  is a quasi-projective and smooth complex variety, whose associated analytic space  $X_{\sigma}^{\rm an}$  is therefore a quasi-projective complex manifold. It is convenient to define  $X^{\rm an}$  as the disjoint union of the  $X_{\sigma}^{\rm an}$ , indexed by  $\sigma$ . For instance, when A is a number field, then  $X^{\rm an}$  is the complex analytic space associated to X as an arithmetic variety over  $\mathbb{Q}$ . Differential geometric objects on  $X^{\rm an}$  such as line bundles, differential forms, metrics, etc. may equivalently be seen as collections of corresponding objects on the  $X_{\sigma}^{\rm an}$ . The complex conjugation induces an anti-holomorphic involution on  $X^{\rm an}$ , and it is customary in Arakelov geometry to impose some compatibility of the analytic data with this action. Let us now recall the definitions of the arithmetic Picard and first Chow groups of X.

#### **Definition 2.1.** A smooth hermitian line bundle on X consists in a pair (L, h), where

- L is a line bundle on X.
- h is a smooth hermitian metric on the holomorphic line bundle  $L^{\rm an}$  on  $X^{\rm an}$  deduced from L, invariant under the action of the complex conjugation. Hence, h is a conjugation invariant collection  $\{h_\sigma\}_{\sigma\colon A\to\mathbb{C}}$ , where  $h_\sigma$  is a smooth hermitian metric on the holomorphic line bundle  $L^{\rm an}_\sigma$  on  $X^{\rm an}_\sigma$  deduced from L by base change and analytification.

The set of isomorphism classes of hermitian line bundles (L,h), with the natural tensor product operation, is a commutative group denoted by  $\widehat{\text{Pic}}(X)$  and called the arithmetic Picard group of X.

**Definition 2.2.** The first arithmetic Chow group  $\widehat{\operatorname{CH}}^1(X)$  of X is the commutative group

• generated by arithmetic divisors, i.e. couples  $(D, g_D)$ , where D is a Weil divisor on X and  $g_D$  is a Green current for the divisor  $D^{\rm an}$ , compatible with complex conjugation. Hence, by definition  $g_D$  is a degree 0 current on  $X^{\rm an}$  that is a  $dd^c$ -potential for the current of integration  $\delta_{D^{\rm an}}$ 

$$dd^c g_D + \delta_{D^{\rm an}} = [\omega_D],$$

up to some smooth differential (1, 1) form  $\omega_D$  on  $X^{an}$ .

• with relations  $(\operatorname{div}(\phi), [-\log|\phi|^2])$ , for non-zero rational functions  $\phi$  on X.

The arithmetic Picard and first Chow groups are related via the first arithmetic Chern class

$$\widehat{\mathrm{c}}_1:\widehat{\mathrm{Pic}}(X)\to\widehat{\mathrm{CH}}^1(X),$$

which maps a hermitian line bundle (L,h) to the class of the arithmetic divisor  $(\operatorname{div}(\ell), [-\log \|\ell\|_h^2])$ , where  $\ell$  is any non-zero rational section of L. This is in fact an isomorphism. We refer the reader to [GS90b, Sec. 2] for a complete discussion.

More generally, Gillet–Soulé developed a theory of arithmetic cycles and Chow rings [GS90a], an arithmetic K-theory and characteristic classes [GS90b, GS90c], and an arithmetic Riemann–Roch theorem [GS92]. While for the comprehension of the theorem below only  $\widehat{CH}^1$ ,  $\widehat{Pic}$  and  $\widehat{c}_1$  are needed, the proof uses all this background, for which we refer to the above references.

Let now  $f: X \to S$  be a smooth projective morphism of arithmetic varieties of relative dimension n, with generic fiber  $X_{\infty}$ . To simplify the exposition, we assume that  $S \to \operatorname{Spec} A$  is surjective and has geometrically connected fibers. In particular, that  $S^{\operatorname{an}}_{\sigma}$  is connected for every embedding  $\sigma$ . More importantly, we suppose that the fibers  $X_s$  are Calabi–Yau, hence they satisfy  $K_{X_s} = \mathscr{O}_{X_s}$ . We define the BCOV line bundle on S as the determinant of cohomology of the virtual vector bundle  $\sum_p (-1)^p p\Omega^p_{X/S}$ , that is, in additive notation for the Picard group of S

(2.6) 
$$\lambda_{BCOV}(X/S) = \sum_{p=0}^{n} (-1)^p p \lambda(\Omega_{X/S}^p) = \sum_{p,q} (-1)^{p+q} p \det R^q f_* \Omega_{X/S}^p.$$

If there is no possible ambiguity, we will sometimes write  $\lambda_{BCOV}$  instead of  $\lambda_{BCOV}(X/S)$ .

For the following statement, we fix an auxiliary conjugation invariant Kähler metric h on  $T_{X^{\mathrm{an}}}$ . We denote by  $\omega$  the associated Kähler form, normalized according to the conventions in Arakelov theory as in (2.1). We assume that the restriction of  $\omega$  to fibers (still denoted by  $\omega$ ) has rational cohomology class. All the  $L^2$  metrics below are computed with respect to  $\omega$  as in (2.2). Depending on the Kähler metric, the line bundle  $\lambda_{BCOV}$  carries a Quillen metric  $h_O$ 

$$h_{Q,s} = T(X_s, \omega) \cdot h_{L^2,s}$$
.

Following [EFiMM18b, Def. 4.1] and [EFiMM18a, Def. 5.2], the Quillen-BCOV metric on  $\lambda_{BCOV}$  is defined by multiplying  $h_Q$  by the correcting factor A in (2.3): for every  $s \in S^{an}$ , we put

$$h_{O,BCOV,s} = A(X_s, \omega) \cdot h_{O,s}$$

It is shown in *loc. cit.* that the Quillen-BCOV is actually a smooth hermitian metric, independent of the choice of  $\omega$ . Besides, according to [EFiMM18b, Def. 5.4] one defines the  $L^2$ -BCOV metric on  $\lambda_{BCOV}$  by

$$(2.7) h_{L^2,BCOV,s} = B(X_s,\omega) \cdot h_{L^2,s},$$

where  $h_{L^2}$  stands for the combination of  $L^2$ -metrics on the Hodge bundles and B was introduced in (2.4). In *loc. cit.* we showed that the function  $s \mapsto B(X_s, \omega)$  is actually locally constant and  $h_{L^2,BCOV}$  is a smooth hermitian metric. Notice that the BCOV invariant defined in (2.5) can then be written as the quotient of the Quillen-BCOV and  $L^2$ -BCOV metrics:

(2.8) 
$$\tau_{BCOV}(X_s) = \frac{h_{Q,BCOV,s}}{h_{L^2,BCOV,s}}.$$

**Theorem 2.3.** Under the above assumptions, there is an equality in  $\widehat{\operatorname{CH}}^1(S)_{\mathbb{Q}} = \widehat{\operatorname{CH}}^1(S) \otimes \mathbb{Q}$ 

(2.9) 
$$\widehat{c}_{1}(\lambda_{BCOV}, h_{Q,BCOV}) = \frac{\chi(X_{\infty})}{12} \widehat{c}_{1}(f_{*}K_{X/S}, h_{L^{2}}).$$

Hence, for any complex embedding  $\sigma$ , any rational section  $\eta$  of  $f_*K_{X/S}$ , any rational section  $\eta_{p,q}$  of  $\det R^q f_*\Omega^p_{X/S}$ , we have an equality of functions on  $S^{\rm an}_{\sigma}$ 

$$(2.10) \qquad \log \tau_{BCOV,\sigma} = \log |\Delta|_{\sigma}^{2} + \frac{\chi(X_{\infty})}{12} \log \|\eta\|_{L^{2},\sigma}^{2} - \sum_{0 \leq p,q \leq n} (-1)^{p+q} p \log \|\eta_{p,q}\|_{L^{2},\sigma}^{2} + \log C_{\sigma},$$

where:

- $\Delta \in K(S)^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ .
- $C_{\sigma} \in \pi^r \mathbb{Q}_{>0}$ , where  $r = \frac{1}{2} \sum (-1)^{k+1} k^2 b_k$  and  $b_k$  is the k-th Betti number of  $X_{\infty}$ .

*Proof.* The proof is a routine application of the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS92, Thm. 7]. We give the details for the convenience of the reader. Consider the virtual vector bundle  $\sum (-1)^p p \Omega_{X/S}^p$ , with virtual hermitian structure deduced from the metric h, and denoted  $h^{\bullet}$ . Its determinant of cohomology  $\lambda_{BCOV}$  carries the Quillen metric  $h_Q$ . The theorem of Gillet–Soulé provides an equality in  $\widehat{CH}^1(S)_{\mathbb{Q}}$ 

$$\widehat{\mathbf{c}}_{1}(\lambda_{BCOV}, h_{Q}) = f_{*}\left(\widehat{\mathbf{ch}}(\sum(-1)^{p} p\Omega_{X/S}^{p}), h^{\bullet})\widehat{\mathrm{Td}}(T_{X/S}, h)\right)^{(1)}$$

$$-a\left(\mathrm{ch}(\sum(-1)^{p} p\Omega_{X^{\mathrm{an}/S^{\mathrm{an}}}}^{p}) \operatorname{Td}(T_{X^{\mathrm{an}/S^{\mathrm{an}}}}) R(T_{X^{\mathrm{an}/S^{\mathrm{an}}}})\right)^{(1)}$$

$$= \frac{1}{12} f_{*}\left(\widehat{\mathbf{c}}_{1}(K_{X/S}, h^{*})\widehat{\mathbf{c}}_{n}(T_{X/S}, h)\right),$$
(2.11)

where  $h^* = (\det h)^{-1}$  is the hermitian metric on  $K_{X/S}$  induced from h. Notice that the topological factor containing the R-genus in loc. cit. vanishes in our situation, since

$$\operatorname{ch}\left(\sum (-1)^{p} p \Omega_{X^{\operatorname{an}}/S^{\operatorname{an}}}^{p}\right) \operatorname{Td}(T_{X^{\operatorname{an}}/S^{\operatorname{an}}}) = -\operatorname{c}_{n-1} + \frac{n}{2} \operatorname{c}_{n} - \frac{1}{12} \operatorname{c}_{1} \operatorname{c}_{n} + \operatorname{higher degree terms}$$

and R has only odd degree terms and  $c_1(T_{X^{\mathrm{an}}/S^{\mathrm{an}}}) = 0$ . Now, the evaluation map  $f^*f_*K_{X/S} \to K_{X/S}$  is an isomorphism, but it is in general not an isometry if we equip  $f_*K_{X/S}$  with the  $L^2$  metric and  $K_{X/S}$  with the metric  $h^*$ . Comparing both metrics yields a relation in  $\widehat{\mathrm{CH}}^1(X)$ 

(2.12) 
$$\widehat{\mathbf{c}}_1(K_{X/S}, h^*) = f^* \widehat{\mathbf{c}}_1(f_* K_{X/S}, h_{L^2}) + [(0, -\log \varphi)].$$

Here  $\varphi$  is the smooth function on  $X^{an}$  given by

$$\varphi = \frac{i^{n^2} \eta \wedge \overline{\eta}}{\|\eta\|_{L^2}^2} \frac{n!}{(2\pi\omega)^n},$$

where  $\eta$  denotes a local trivialization of  $f_*K_{X^{\mathrm{an}}/S^{\mathrm{an}}}$ , thought of as a section of  $K_{X^{\mathrm{an}}/S^{\mathrm{an}}}$  via the evaluation map. Multiplying (2.12) by  $\widehat{\mathbf{c}}_n(T_{X/S},h)$  and applying  $f_*$  and the projection formula for arithmetic Chow groups, we find

$$f_* \left( \widehat{\mathbf{c}}_1(K_{X/S}, h^*) \widehat{\mathbf{c}}_n(T_{X/S}, h) \right) = f_* \left( f^* \widehat{\mathbf{c}}_1(f_* K_{X/S}, h_{L^2}) \widehat{\mathbf{c}}_n(T_{X/S}, h) \right) + f_* \left( [(0, -\log \varphi)] \widehat{\mathbf{c}}_n(T_{X/S}, h) \right)$$

$$= \chi(X_{\infty}) \widehat{\mathbf{c}}_1(f_* K_{X/S}, h_{L^2}) + \left[ \left( 0, -\int_{X^{\mathrm{an}}/S^{\mathrm{an}}} (\log \varphi) \widehat{\mathbf{c}}_n(T_{X^{\mathrm{an}}/S^{\mathrm{an}}}, h) \right) \right],$$

where  $c_n(T_{X/S}, h)$  is the n-th Chern–Weil differential form of  $(T_{X^{an}/S^{an}}, h)$ . Together with (2.11), this shows that the metric

$$h_{Q,BCOV} = h_Q \cdot \exp\left(-\frac{1}{12} \int_{X^{\mathrm{an}}/S^{\mathrm{an}}} (\log \varphi) \mathbf{c}_n(T_{X^{\mathrm{an}}/S^{\mathrm{an}}}, h)\right)$$

indeed satisfies (2.9).

The outcome (2.10) is a translation of the meaning of the equality (2.9) in  $\widehat{\operatorname{CH}}^1(S)_{\mathbb Q}$ , in terms of the constructions (2.8) and (2.7). By [EFiMM18a, Prop. 4.2] the normalizing factor B is constant on each connected manifold  $S^{\operatorname{an}}_{\sigma}$  and would be rational if the  $L^2$  inner products on cohomology groups were computed with  $h/2\pi$ . With this understood, we find

(2.13) 
$$\operatorname{vol}_{L^{2}}(H^{k}(X_{s},\mathbb{Z}),\omega) \in (2\pi)^{-kb_{k}} \mathbb{Q}_{>0}^{\times}$$

for any  $s \in S_{\sigma}^{an}$ . Together with the definition of B (2.4), this is responsible for the constants  $C_{\sigma}$ .

- Remark 2.4. (1) The use of the arithmetic Riemann–Roch theorem requires an algebraic setting, but directly yields the existence of the rational function  $\Delta$ . By contrast, previous techniques (cf. *e.g.* [FLY08, Sections 7 & 10]) rely on subtle integrability estimates of the functions in (2.10), in order to ensure that the *a priori* pluriharmonic function  $\log |\Delta|_{\sigma}^2$  is indeed the logarithm of a rational function. The arithmetic Riemann–Roch theorem further provides the field of definition of  $\Delta$  and the constants  $C_{\sigma}$ .
  - (2) In the case of a Calabi–Yau 3-fold defined over a number field, similar computations were done by Maillot–Rössler [MR12, Sec. 2].
- 2.4. **Kronecker limit formulas for families of Calabi–Yau hypersurfaces.** In this section we give an example of use of Theorem 2.3 and we determine the BCOV invariant for families of Calabi–Yau hypersurfaces in Fano manifolds. The argument provides a simplified model for the later computation of the BCOV invariant of the family of the mirror Calabi–Yau hypersurfaces.

Let V be a complex Fano manifold, with very ample anti-canonical bundle  $-K_V$ . We consider the anti-canonical embedding of V into  $|-K_V| = \mathbb{P}(H^0(V, -K_V)) \simeq \mathbb{P}^N$ , whose smooth hyperplane sections are Calabi–Yau manifolds. The dual projective space  $\check{\mathbb{P}} = \mathbb{P}(H^0(V, -K_V)^\vee) \simeq \check{\mathbb{P}}^N$  parametrizes hyperplane sections, and contains an irreducible subvariety  $\Delta \subseteq \check{\mathbb{P}}$  which corresponds to singular such sections [GKZ08, Chap. 1, Prop. 1.3]. We assume that  $\Delta$  is a hypersurface in  $\check{\mathbb{P}}$ . This is in general not true, and a necessary condition is proven in [GKZ08, Chap. 1, Cor. 1.2]. Denote by U the quasi-projective complement  $U := \check{\mathbb{P}} \setminus \Delta$ . Denote by  $f: \mathscr{X} \to \check{\mathbb{P}}$  the universal family of hyperplane sections. Therefore f is smooth U, and the corresponding BCOV line bundle  $\lambda_{BCOV}$  is thus defined on U.

**Lemma 2.5.** For some positive integer m, the line bundles  $(f_*K_{\mathcal{X}/U})^{\otimes m}$  and  $\lambda_{BCOV}^{\otimes m}$  have trivializing sections. These are unique up to constants.

*Proof.* A standard computation shows that  $\text{Pic}(U) = \mathbb{Z}/\deg \Delta$ , providing the first claim of the lemma. For the second assertion, for any of the line bundles under consideration, let  $\theta$  and  $\theta'$  be two trivializations on U. Therefore,  $\theta = h\theta'$  for some invertible function h on U. The previous description of Pic(U) shows that the divisor of h, as a rational function on  $\check{\mathbb{P}}$ , is supported on  $\Delta$ . As  $\Delta$  is irreducible, in the projective space  $\check{\mathbb{P}}$  this is only possible if the divisor vanishes. We conclude that h is necessarily constant.

For the following statement, we need a choice of auxiliary Kähler metric on  $\mathcal{X}$  (restricted to U), whose Arakelov theoretic Kähler form has fiberwise rational cohomology class. We compute  $L^2$  norms on Hodge bundles and on  $\lambda_{BCOV}$  with respect to this choice.

**Theorem 2.6.** For some integer m > 0 as in the lemma, let  $\beta$  be a trivialization of  $\lambda_{BCOV}^{\otimes m}$  and  $\eta$  a trivialization of  $(f_*K_{\mathcal{X}/U})^{\otimes m}$ . Then there is a global constant C such that, for any Calabi–Yau hyperplane section  $X_H = V \cap H$ , we have

$$\tau_{BCOV}(X_H) = C \|\eta\|_{L^2}^{\chi/6m} \|\beta\|_{L^2}^{-2/m}.$$

*Proof.* We apply Theorem 2.3 to  $f: \mathcal{X} \to U$  (over  $\mathbb{C}$ ), which in terms of  $\beta$  and  $\eta$  becomes

$$m \log \tau_{BCOV}(X_H) = \log|g|^2 + \frac{\chi}{12} \log \|\eta\|_{L^2}^2 - \log \|\beta\|_{L^2}^2 + \log C$$

for some regular invertible function g on U and some constant C. By construction, as a rational function on  $\check{\mathbb{P}}$ , g must have its zeros or poles along  $\Delta$ . Since  $\Delta$  is irreducible this forces g to be constant.

- *Remark* 2.7. (1) When V is a toric variety with very ample anti-canonical class, all of the constructions can in fact be done over the rational numbers. The sections  $\beta$  and  $\eta$  can be taken to be defined over  $\mathbb{Q}$ , and unique up to a rational number. With this choice, the constant C takes the form stated in Theorem 2.3.
  - (2) In the case when the discriminant  $\Delta$  has higher codimension, we have  $\operatorname{Pic}(U) \simeq \operatorname{Pic}(\check{\mathbb{P}})$ . In particular,  $\lambda_{BCOV}$  uniquely extends to a line bunlde  $\check{\mathbb{P}}$ . The existence of the canonical (up to constant) trivializations  $\beta$  and  $\eta$  is no longer true. However, one can propose a variant of the theorem where  $\beta$  and  $\eta$  are trivializations outside a chosen ample divisor in  $\check{\mathbb{P}}$ .
    - 3. THE DWORK AND MIRROR FAMILIES, AND THEIR HODGE BUNDLES
- 3.1. **The mirror family and its crepant resolution.** We review general facts on the Dwork pencil of Calabi–Yau hypersurfaces, and the construction of the mirror of Calabi–Yau hypersurfaces in projective space. Initially, we work over the field of complex numbers. Rationality refinements will be made along the way.

Let  $n \ge 4$  be an integer. The Dwork pencil  $\mathscr{X} \to \mathbb{P}^1$  is defined by the hypersurface of  $\mathbb{P}^n \times \mathbb{P}^1$  of equation

$$F_{\psi}(x_0,\ldots,x_n) := \sum_{j=0}^n x_j^{n+1} - (n+1)\psi x_0 \ldots x_n = 0, \quad [x_0:x_1:\ldots:x_n] \in \mathbb{P}^n, \quad \psi \in \mathbb{P}^1.$$

The smooth fibers of this family are Calabi–Yau manifolds of dimension n-1. The singular fibers are:

- fiber at  $\psi = \infty$ , given by the divisor with normal crossings  $x_0 \cdot ... \cdot x_n = 0$ .
- the fibers where  $\psi^{n+1} = 1$ . These fibers have ordinary double point singularities. The singular points have projective coordinates  $(x_0, ..., x_n)$  with  $x_0 = 1$  and  $x_j^{n+1} = 1$  for all  $j \ge 1$ , and  $\prod_i x_i = \psi^{-1}$ .

Denote by  $\mu_{n+1}$  the group of the (n+1)-th roots of unity. Let K be the kernel of the multiplication map  $\mu_{n+1}^{n+1} \to \mu_{n+1}$ . Let also  $\Delta$  be the diagonal embedding of  $\mu_{n+1}$  in K. The group  $G := K/\Delta$  acts naturally on the fibers  $X_{\psi}$  of  $\mathscr{X} \to \mathbb{P}^1$  by multiplication of the projective coordinates, and we denote the quotient space by  $\mathscr{Y} \to \mathbb{P}^1$ .

We notice that the above construction can be done over  $\mathbb{Q}$ . Indeed,  $F_{\psi}$  is already defined over  $\mathbb{Q}$ , and the groups K,  $\Delta$  are finite algebraic groups over  $\mathbb{Q}$ , and hence so does the quotient G. The action of G on  $F_{\psi}$  is defined over  $\mathbb{Q}$  as well, as one can see by examining the compatibility with the action of  $\operatorname{Aut}(\mathbb{C}/\mathbb{Q})$  on the  $\mathbb{C}$  points of  $\mathscr{X}$ , or alternatively by writing the co-action at the level of algebras. Therefore, the quotient  $\mathscr{Y} = \mathscr{X}/G$  is defined over  $\mathbb{Q}$ , and so does the projection map  $\mathscr{Y} \to \mathbb{P}^1$ .

**Lemma 3.1.** The total space of the restricted family  $\mathscr{Y} \to \mathbb{A}^1$  has rational Gorenstein singularities. It has a relative canonical line bundle  $K_{\mathscr{Y}/\mathbb{A}^1}$ , obtained by descent from  $K_{\mathscr{X}/\mathbb{A}^1}$ .

*Proof.* To lighten notations, let us write in this proof  $\mathscr X$  and  $\mathscr Y$  for the corresponding restrictions to  $\mathbb A^1$ . The total space  $\mathscr X$  is non-singular, and  $\mathscr Y$  is a quotient of it by the action of a finite group. Therefore,  $\mathscr Y$  has rational singularities. In particular, it is normal and Cohen–Macaulay. Consequently, if  $\mathscr Y^{ns}$  is the non-singular locus of  $\mathscr Y$ , and  $j\colon \mathscr Y^{ns}\hookrightarrow \mathscr Y$  the open immersion, then we have a relation between relative dualizing sheaves  $j_*\omega_{\mathscr Y^{ns}/\mathbb A^1}=\omega_{\mathscr Y/\mathbb A^1}$ . We will use this below.

Now for the Gorenstein property and the descent claim. Notice that since  $\mathbb{A}^1$  is non-singular,  $\mathscr{Y}$  is Gorenstein if, and only if, the fibers of  $\mathscr{Y} \to \mathbb{A}^1$  are Gorenstein. We will implicitly confound both the absolute and relative points of view. We introduce  $\mathscr{X}^{\circ}$  the complement of the fixed locus of G, and  $\mathscr{X}^*$  the smooth locus of  $\mathscr{X} \to \mathbb{A}^1$ . These are G-invariant open subschemes of  $\mathscr{X}$  and constitute an open cover, because the ordinary double points in the fibers of  $\mathscr{X} \to \mathbb{A}^1$  are disjoint from the fixed point locus of G. Then  $\mathscr{Y}^{\circ} = \mathscr{X}^{\circ}/G$  and  $\mathscr{Y}^* = \mathscr{X}^*/G$  form an open cover of  $\mathscr{Y}$ , and it is enough to proceed for each one separately.

Since G acts freely on  $\mathscr{X}^{\circ}$ , the quotient  $\mathscr{Y}^{\circ}$  is non-singular, and is therefore Gorenstein. The morphism  $\mathscr{X}^{\circ} \to \mathscr{Y}^{\circ}$  is étale, and hence  $K_{\mathscr{X}^{\circ}/\mathbb{A}^{1}}$  descends to  $K_{\mathscr{Y}^{\circ}/\mathbb{A}^{1}}$ .

For  $\mathscr{Y}^*$ , we observe that G preserves a relative holomorphic volume form on  $\mathscr{X}^*$ . Indeed, in affine coordinates  $z_k = \frac{x_k}{x_i}$  on the open set  $x_j \neq 0$ , and where  $\partial F_{\psi}/\partial z_i \neq 0$ , the expression

(3.1) 
$$\theta_0 = \frac{(-1)^{i-1} dz_0 \wedge \dots \widehat{dz_i} \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n}{\partial F_{\psi} / \partial z_i} \Big|_{F_{\psi} = 0}$$

provides such an invariant relative volume form. This entails that  $K_{\mathcal{X}^*/\mathbb{A}^1}$  descends to an invertible sheaf  $\mathcal{K}$  on  $\mathcal{Y}^*$ . Now, the singular locus of  $\mathcal{Y}^*$  is contained in the image of the fixed point set of G on  $\mathcal{X}^*$ . We infer that  $\mathcal{K}$  is an invertible extension of the relative canonical bundle of  $(\mathcal{Y}^*)^{ns} \to \mathbb{A}^1$ . But  $\mathcal{Y}^*$  is normal so that  $\mathcal{K} \simeq j_* j^* \mathcal{K}$ . Then as mentioned at the beginning of the proof,  $j_* \omega_{\mathcal{Y}^{ns}/\mathbb{A}^1} = \omega_{\mathcal{Y}/\mathbb{A}^1}$  and we conclude, since  $\mathcal{K}$  is also an extension of  $\omega_{\mathcal{Y}^{ns}/\mathbb{A}^1}$ .

Because the BCOV invariant has not been fully developed for Calabi–Yau orbifolds (see nevertheless [Yos17] for some three-dimensional cases), we need crepant resolutions of the varieties  $Y_{\psi}$ . This needs to be done in families, so that the results of §2.3 apply. The family of crepant resolutions  $\mathcal{Z} \to \mathbb{P}^1$  that we exhibit will be called the mirror family, although it is not unique. We also have to address the rationality of the construction.

**Lemma 3.2.** There is a projective birational morphism  $\mathcal{Z} \to \mathcal{Y}$  of algebraic varieties over  $\mathbb{Q}$ , such that

- (1)  $\mathcal{Z}$  is smooth.
- (2) If  $\psi^{n+1} = 1$ , the fiber  $Z_{\psi}$  has a single ordinary double point singularity.
- (3) If  $\psi = \infty$ ,  $Z_{\infty}$  is a simple normal crossings divisor in  $\mathcal{Z}$ .

- (4) Otherwise,  $Z_{\psi} \to Y_{\psi}$  is a crepant resolution of singularities. In particular,  $Z_{\psi}$  is a smooth Calabi–Yau variety.
- (5) The smooth complex fibers  $Z_{\psi}$  are mirror to the  $X_{\psi}$ , in that their Hodge numbers satisfy  $h^{p,q}(Z_{\psi}) = h^{n-1-p,q}(X_{\psi})$ . In particular, the smooth  $Z_{\psi}$  are Calabi–Yau with  $\chi(Z_{\psi}) = (-1)^{n-1}\chi(X_{\psi})$ .

*Proof.* The proof of (1)–(4) is based on [DHZ98, Sec. 8 (v)], [DHZ06] and [BG14, Prop. 3.1], together with Hironaka's resolution of singularities. We recall the strategy, in order to justify the existence of a model over  $\mathbb{Q}$ .

Introduce  $W = \mathbb{P}^n/G$ . We claim this is a split toric variety over  $\mathbb{Q}$ . First of all, it can be realized as the hypersurface in  $\mathbb{P}^{n+1}_{\mathbb{Q}}$  of equation

$$W \colon y_0^{n+1} = \prod_{j=1}^{n+1} y_j.$$

Second, the associated torus is split over  $\mathbb{Q}$ . It is actually given by  $\mathbb{G}_{m\ \mathbb{Q}}\times\mathbb{T}$ , where  $\mathbb{T}$  is the kernel of the multiplication map  $\mathbb{G}_{m\ \mathbb{Q}}^{n+1}\to\mathbb{G}_{m\ \mathbb{Q}}$ . Finally, the action of the torus on W is defined over  $\mathbb{Q}$ :

$$\big((t_0,t_1,\ldots,t_{n+1}),(y_0,y_1,\ldots,y_{n+1})\big)\mapsto (t_0y_0,t_0t_1y_1,\ldots,t_0t_{n+1}y_{n+1}).$$

Once we know that W is a split toric variety over  $\mathbb{Q}$ , with same equation as in [DHZ06, Application 5.5], the toric and crepant projective resolution exhibited in *loc. cit.* automatically works over  $\mathbb{Q}$  as well. We write  $\widetilde{W}$  for this resolution of W.

We now consider  $\mathscr{Y}$  as a closed integral  $\mathbb{Q}$ -subscheme of  $\widetilde{W} \times \mathbb{P}^1$ . Let  $\widetilde{\mathscr{Y}}$  be the strict transform of  $\mathscr{Y}$  in  $\widetilde{W} \times \mathbb{P}^1$ . By [DHZ98, Sec. 8 (v)], the fibers of  $\widetilde{\mathscr{Y}}$  at  $\psi \in \mathbb{C} \setminus \mu_{n+1}$  are projective crepant resolutions of the fibers  $Y_{\psi}$ . In particular,  $\widetilde{\mathscr{Y}}$  is smooth over  $\mathbb{C} \setminus \mu_{n+1}$ , and in turn this implies smoothness over the complement U of the closed subscheme  $V(\psi^{n+1}-1)$  of  $\mathbb{A}^1_{\mathbb{Q}}$ . Necessarily, the fibers of  $\widetilde{\mathscr{Y}}$  over U have trivial canonical bundle as well. For the fibers at  $\psi^{n+1}=1$ , the claim of the lemma requires two observations:

- (1) the ordinary double points of  $X_{\psi}$  are permuted freely and transitively by G, and get identified to a single point in the quotient  $Y_{\psi}$ . This entails that the total space  $\mathscr{Y}$  is non-singular in a neighborhood of these points, and that they remain ordinary double points of  $\mathscr{Y} \to \mathbb{P}^1$ .
- (2) the center of the toric resolution is disjoint from the ordinary double points, since it is contained in the locus of  $\mathbb{P}^n/G$  where two or more projective coordinates vanish. Therefore, the morphism  $\mathcal{Z} \to \mathcal{Y}$  is an isomorphism in a neighborhood of these points. Finally, on the complement,  $Z_{\psi}$  is a resolution of singularities of  $Y_{\psi}$ . Indeed, this is a local question in a neighborhood of the fixed points of G, so that the above references [DHZ98, DHZ06] still apply.

Finally,  $\widetilde{\mathscr{Y}}$  is by construction smooth on the complement of the fiber  $\psi = \infty$ . After a resolution of singularities given by blowups with smooth centers in  $\widetilde{Y}_{\infty}$  (defined over  $\mathbb{Q}$ ), we obtain a smooth algebraic variety  $\mathscr{Z}$  over  $\mathbb{Q}$ , such that  $Z_{\infty}$  is a simple normal crossings divisor in  $\mathscr{Z}$ . This sets items (1)–(4).

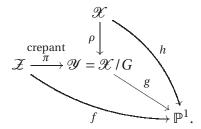
For (5), we refer for instance to [BD96, Thm. 6.9, Conj. 7.5 & Ex. 8.7]. This is specific to the Dwork pencil. More generally, we can cite work of Yasuda, who proves an invariance property of orbifold Hodge structures (and hence orbifold Hodge numbers) under crepant resolutions, for quotient Gorenstein singularities [Yas04, Thm. 1.5]. Orbifold Hodge numbers coincide with

stringy Hodge numbers of global (finite) quotient orbifolds, whose underlying group respects a holomorphic volume form [BD96, Thm. 6.14]. Finally, by [BB96, Thm. 4.15], stringy Hodge numbers satisfy the expected mirror symmetry property for the mirror pairs constructed by Batyrev [Bat94].

**Definition 3.3.** The point  $\infty \in \mathbb{P}^1$  is called the MUM point of the family  $f: \mathcal{Z} \to \mathbb{P}^1$ . The points  $\xi \in \mathbb{P}^1$  with  $\xi^{n+1} = 1$  are called the ODP points.

The terminology MUM stands for *maximally unipotent Monodromy*, and it will be justified later in Lemma 4.1. The terminology ODP stands for *Ordinary Double Point*.

3.2. **Comparison of Hodge bundles.** Recall from the previous subsection the families  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$ , fibred over  $\mathbb{P}^1$ :



We denote by U the maximal Zariski open subset of  $\mathbb{P}^1$  where f (resp. h) is smooth. When it is clear from the context, we will still write  $\mathscr{X}$ ,  $\mathscr{Y}$  and  $\mathscr{Z}$  for the total spaces of the fibrations restricted to U. Otherwise, we add an index U to mean the restriction to U. We let  $\mathscr{Y}^{\circ}$  be the non-singular locus of  $\mathscr{Y}_U$ . It is the étale quotient of  $\mathscr{X}^{\circ}$ , the complement in  $\mathscr{X}_U$  of the fixed point set of G. They are both open subsets whose complement has codimension  $\geq 2$ .

We begin our considerations by working complex analytically. Our discussion is based on a minor adaptation of [Ste77, Sec. 1] to the relative setting. First of all, we observe that the higher direct images  $R^k g_* \mathbb{C}$  are locally constant sheaves, and actually  $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$ . Indeed, we have the equality  $\mathbb{C}_{\mathscr{Y}} = (\rho_* \mathbb{C}_{\mathscr{X}})^G$ . Moreover, since G is finite, so is  $\rho$  and taking G-invariants is an exact functor in the category of sheaves of  $\mathbb{C}[G]$ -modules. A spectral sequence argument allows us to conclude. Similarly, one has  $R^k g_* \mathbb{Q} \simeq (R^k h_* \mathbb{Q})^G$ .

allows us to conclude. Similarly, one has  $R^kg_*\mathbb{Q}\simeq (R^kh_*\mathbb{Q})^G$ . Let now  $\widetilde{\Omega}^{\bullet}_{\mathscr{Y}/U}$  be the relative holomorphic de Rham complex of  $\mathscr{Y}\to U$ , in the orbifold sense. It is constructed as follows. If  $j\colon \mathscr{Y}^\circ\hookrightarrow \mathscr{Y}_U$  is the open immersion, then we let  $\widetilde{\Omega}^{\bullet}_{\mathscr{Y}_U}:=j_*\Omega^{\bullet}_{\mathscr{Y}^\circ}$ , and we derive the relative version  $\widetilde{\Omega}^{\bullet}_{\mathscr{Y}/U}$  out of it in the usual manner. An equivalent presentation is

$$\widetilde{\Omega}_{\mathcal{Y}/U}^{\bullet} = (\rho_* \Omega_{\mathcal{X}/U}^{\bullet})^G.$$

The complex  $\widetilde{\Omega}_{\mathscr{Y}/U}^{\bullet}$  is a resolution of  $g^{-1}\mathscr{O}_{U}$ . Hence its k-th relative hypercohomology computes  $(R^{k}g_{*}\mathbb{C})\otimes\mathscr{O}_{U}$ , and satisfies

$$(3.2) R^k g_* \widetilde{\Omega}_{\mathcal{U}/U}^{\bullet} \simeq (R^k h_* \Omega_{\mathcal{X}/U}^{\bullet})^G,$$

compatibly with  $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$ . It has a Hodge filtration and Gauss–Manin connection defined in the usual way, satisfying a relationship analogous to (3.2). Equipped with this extra structure,  $R^k g_* \mathbb{Q}$  defines a variation of pure rational Hodge structures of weight k.

The canonical identification  $\widetilde{\Omega}_{\mathcal{Y}_U}^{\bullet} = \pi_* \Omega_{\mathcal{Z}_U}^{\bullet}$  established in [Ste77, Lemma 1.11] induces a natural morphism

(3.3) 
$$\widetilde{\Omega}_{\mathscr{Y}/U}^{\bullet} \longrightarrow \pi_{*}(\Omega_{\mathscr{Z}/U}^{\bullet}).$$

The restriction of (3.3) to  $\mathscr{Y}^{\circ}$  is given by pull-back of differential forms. We derive a natural map

$$(3.4) (R^k h_* \Omega_{\mathcal{X}/U}^{\bullet})^G \simeq R^k g_* \widetilde{\Omega}_{\mathcal{Y}/U}^{\bullet} \longrightarrow R^k f_* \Omega_{\mathcal{X}/U}^{\bullet},$$

which is an injective morphism of variations of pure Hodge structures of weight k, cf. [Ste77, Cor. 1.5]. It is in particular compatible with restriction to the fibers, and remains injective on those. It can be checked to be compatible with the Q-structures.

The following lemma summarizes the compatibility of (3.4) with the cup-product between Hodge bundles of complementary bi-degree. Before the statement, we recall from Lemma 3.1 that  $\mathscr{Y}_U$  is Gorenstein, and  $K_{\mathscr{X}/U}$  descends to the relative canonical bundle  $K_{\mathscr{Y}/U}$ .

Lemma 3.4.

**nma 3.4.** (1)  $\widetilde{\Omega}^n_{\mathscr{Y}/U}$  is the relative canonical bundle  $K_{\mathscr{Y}/U}$ . (2) The natural morphism  $R^k g_* \widetilde{\Omega}^{\bullet}_{\mathscr{Y}/U} \longrightarrow R^k f_* \Omega^{\bullet}_{\mathscr{Z}/U}$  induces a commutative diagram

$$R^{q}g_{*}\widetilde{\Omega}^{p}_{\mathcal{Y}/U}\otimes R^{n-q}g_{*}\widetilde{\Omega}^{n-p}_{\mathcal{Y}/U}\longrightarrow R^{n}g_{*}K_{\mathcal{Y}/U}$$

$$\downarrow \qquad \qquad \qquad \downarrow \text{tr}$$

$$R^{q}f_{*}\Omega^{p}_{\mathcal{Z}/U}\otimes R^{n-q}f_{*}\Omega^{n-p}_{\mathcal{Z}/U}\longrightarrow R^{n}f_{*}K_{\mathcal{Z}/U}$$

(3) The natural isomorphism  $R^k g_* \widetilde{\Omega}^{\bullet}_{\mathscr{Y}/U} \simeq (R^k h_* \Omega^{\bullet}_{\mathscr{X}/U})^G$  induces a commutative diagram

$$R^{q}g_{*}\widetilde{\Omega}^{p}_{\mathscr{Y}/U}\otimes R^{n-q}g_{*}\widetilde{\Omega}^{n-p}_{\mathscr{Y}/U}\longrightarrow R^{n}g_{*}K_{\mathscr{Y}/U}\xrightarrow{\operatorname{tr}}\mathscr{O}_{U}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow |G|.$$

$$R^{q}h_{*}\Omega^{p}_{\mathscr{X}/U}\otimes R^{n-q}h_{*}\Omega^{n-p}_{\mathscr{X}/U}\longrightarrow R^{n}h_{*}K_{\mathscr{X}/U}\xrightarrow{\operatorname{tr}}\mathscr{O}_{U}$$

*Proof.* For the first property, we notice that  $\rho^* K_{\mathscr{Y}/U} = K_{\mathscr{X}/U}$ , since both coincide outside a codimension  $\geq 2$  closed subset and  $\mathcal{X}_{II}$  is smooth. Then we have the string of equalities

$$\widetilde{\Omega}^n_{\mathcal{Y}/U} = \rho_*(K_{\mathcal{X}/U})^G = (K_{\mathcal{Y}/U} \otimes \rho_* \mathcal{O}_{\mathcal{X}_U})^G = K_{\mathcal{Y}/U} \otimes (\rho_* \mathcal{O}_{\mathcal{X}_U})^G = K_{\mathcal{Y}/U}.$$

For the first diagram, only the commutativity of the triangle requires a justification. It is a consequence of the three following facts: i) the transitivity of trace maps with respect to composition of morphisms [Har66, Thm. 10.5 (TRA1)]; ii) the crepant resolution property  $\pi^* K_{\mathscr{Y}/U} = K_{\mathscr{Z}/U}$ and iii)  $\mathcal{Y}_U$  has rational singularities, so that  $R\pi_*\mathcal{O}_{\mathcal{X}_U} = \mathcal{O}_{\mathcal{Y}_U}$ . The argument is similar for the second diagram. Briefly, one combines: i) the transitivity of trace maps; ii) the duality  $\rho_*K_{\mathcal{X}_U/\mathcal{Y}_U}=$  $\operatorname{Hom}_{\mathscr{O}_{\mathscr{Y}_U}}(\rho_*\mathscr{O}_{\mathscr{X}_U},\mathscr{O}_{\mathscr{Y}_U})$  and iii) the trace  $\operatorname{tr}\colon \rho_*K_{\mathscr{X}_U/\mathscr{Y}_U}\to\mathscr{O}_{\mathscr{Y}_U}$  is given by  $\varphi\mapsto \varphi(1)$  [Har66, proof of Prop. 6.5], and the composite map

$$K_{\mathcal{Y}/U} \longrightarrow \rho_* K_{\mathcal{X}/U} = K_{\mathcal{Y}/U} \otimes \rho_* K_{\mathcal{X}_U/\mathcal{Y}_U} \overset{\mathrm{id} \otimes \mathrm{tr}}{\longrightarrow} K_{\mathcal{Y}/U}$$

is the multiplication by |G|. This is clear over  $\mathscr{Y}^{\circ}$ , since it is the étale quotient of  $\mathscr{X}^{\circ}$  by G. It is then necessarily true everywhere.

In the case of direct images of relative canonical sheaves, the discussion above specializes to the chain of isomorphism of line bundles

$$(3.5) (h_*K_{\mathcal{X}/U})^G \xrightarrow{\sim} g_*K_{\mathcal{Y}/U} \xrightarrow{\sim} f_*K_{\mathcal{Z}/U}.$$

Moreover, these are the natural morphisms already defined over Q. We leave the details to the reader.

Remark 3.5. In contrast to (3.5), it is in general not true that the injective morphism  $R^kg_*\widetilde{\Omega}^{\bullet}_{\mathscr{Y}/IJ} \to \mathbb{R}^{-1}$  $R^k f_* \Omega^{\bullet}_{\mathcal{Z}/U}$  is an isomorphism of variations of Hodge structures. The fibers of  $R^k g_* \widetilde{\Omega}^{\bullet}_{\mathscr{Y}/U}$  are but a piece of so-called orbifold cohomology groups, which also includes the cohomology of the so-called twisted sectors. It is known that the orbifold cohomology of a proper variety with quotient Gorenstein singularities is isomorphic, as a Hodge structure, to the cohomology of a crepant resolution [Yas04, Thm. 1.5]. The isomorphism is however not explicit. In any event, loc. cit. relates the Hodge numbers of both structures (see the proof of Lemma 3.2).

For later use, we record the following lemma.

**Lemma 3.6.** Let  $h^{p,q}$  be the rank of the Hodge bundle  $R^q f_* \Omega^p_{_{\mathcal{I}/II}}$ . Then:

- $\begin{array}{l} \bullet \ \, h^{p,q} = 1 \ \, if \, p+q = n-1 \, \, and \, \, p \neq q. \\ \bullet \ \, h^{p,p} = \sum_{j=0}^p (-1)^j \binom{n+1}{j} \binom{(p+1-j)n+p}{n} + \delta_{2p,n-1}. \end{array}$
- $h^{p,q} = 0$  otherwise.

In particular,

$$\chi(Z_{\psi}) = (-1)^{n-1} \chi(X_{\psi}) = (-1)^{n-1} \left( \frac{(-n)^{n+1} - 1}{n+1} + n + 1 \right).$$

Moreover, in even relative dimension n-1=2d and for any choice of polarization, the sheaf  $(R^d f_* \Omega^d_{\mathcal{I}/U})_{\text{prim}}$  has rank one.

*Proof.* The items are a consequence of the mirror symmetry property for the Hodge numbers in Lemma 3.2, and the computation of the cohomology of a hypersurface in projective space (cf. [BD96, Ex. 8.7]). The claim on primitive cohomology follows from the primitive decomposition together with the formula for all the  $h^{p,p}$ ,  $p \le d$ .

3.3. Sections  $\eta_k$  of the middle degree Hodge bundles. We maintain the setting and notations of the previous subsection. We now further compare the middle degree Hodge bundles of the Dwork family  $h: \mathcal{X} \to U$  and that of its mirror  $f: \mathcal{Z} \to U$ , by constructing explicit sections via Griffiths' residue method [Gri69]. We introduce primitivity notions for the relative Hodge bundles, induced by any projective factorization of f and the natural projective embedding of h. Observe the latter is G-equivariant and defined over  $\mathbb{Q}$ . We also require the polarization for  $\mathcal{Z} \to U$  to be defined over  $\mathbb{Q}$ . Then the primitive Hodge bundles are defined in the category of Q-schemes.

Our reasoning starts in the complex analytic category. Denote by  $H = x_0 \cdot x_1 \cdot ... \cdot x_n$  and  $\Omega =$  $\sum (-1)^i x_i dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \in H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(n+1))$ . For  $\psi \in U(\mathbb{C})$ , the residue along  $X_{\psi}$ 

$$\theta_k = \operatorname{res}_{X_{\psi}} \left( \frac{k! H^k \Omega}{F_{\psi}^{k+1}} \right)$$

defines a G-invariant element of  $H^{n-1}(X_{\psi})$ , still denoted  $\theta_k$ . For k=0, this indeed agrees with the holomorphic volume form (3.1). By [Gri69, Thm. 8.3], we actually have for k = 0, ..., n-1

$$\theta_k \in F^{n-1-k} H^{n-1}(X_{\psi})_{\text{prim}}^G \setminus F^{n-k} H^{n-1}(X_{\psi})_{\text{prim}}^G.$$

Taking into account the injective morphism (3.4) and the Hodge numbers computed in Lemma 3.6, we see that  $H^{n-1-k,k}(X_{\psi})^G_{\text{prim}}$  is necessarily one-dimensional and  $\theta_k$  projects to a basis element of it. Let us call  $\sigma_k$  this projected element. In the families setting,  $\theta_k$  and  $\sigma_k$  define sections of the corresponding Hodge bundles. In a nutshell, the collection of sections  $\theta_k$  forms a basis of  $(R^{n-1}h_*\Omega^{\bullet}_{\mathcal{X}/U})^G_{\text{prim}}$  adapted to the Hodge filtration, and each individual  $\sigma_k$  is a trivialization of  $(R^kh_*\Omega^{n-1-k}_{\mathcal{X}/U})^G_{\text{prim}}$ .

Now let  $\nabla$  be the Gauss–Manin connection acting on  $R^{n-1}h_*\Omega^{\bullet}_{\mathcal{X}/U}$ . It is compatible with the G-action after §3.2, and it preserves primitive classes as well. From the definition of the sections  $\theta_k$ , one can check the following recurrence:

(3.6) 
$$\nabla_{d/d\psi} \theta_k = \operatorname{res}_{X_{\psi}} \left( \frac{\partial}{\partial \psi} \left( \frac{k! H^k \Omega}{F_{\psi}^{k+1}} \right) \right) = (n+1)\theta_{k+1}.$$

Notice that  $\theta_0$  lies in the *G*-invariant primitive cohomology for obvious reasons, since it belongs to  $(h_*K_{\mathcal{X}/U})^G$ . Therefore this recurrence explains that all the  $\theta_k$  are *G*-invariant and primitive as well. This argument is at the basis of the following proposition:

**Proposition 3.7.** The natural morphism (3.4) induces an isomorphism of variations of Hodge structures

$$(3.7) (R^{n-1}h_*\Omega^{\bullet}_{\mathcal{X}/U})^G_{\text{prim}} \xrightarrow{\sim} (R^{n-1}f_*\Omega^{\bullet}_{\mathcal{X}/U})_{\text{prim}}.$$

*Proof.* By the Hodge numbers computed in Lemma 3.6, it is enough to check that the  $\theta_k$  are mapped into primitive classes. Let  $\theta_k'$  be the image of  $\theta_k$  under (3.4). As (3.4) is compatible with Gauss-Manin connections, the  $\theta_k'$  satisfy the analogous recurrence to (3.6). Because  $f_*K_{\mathcal{Z}/U}$  is primitive and the Gauss-Manin connection preserves primitive cohomology, we see that the  $\theta_k'$  land in the primitive cohomology.

- Remark 3.8. (1) Notice that there was no *a priori* compatibility between the primitivity notions for  $\mathscr{X} \to U$  and  $\mathscr{Z} \to U$ . The proposition depends crucially on the concrete geometrical setting and Hodge structures.
  - (2) Although (3.7) is an isomorphism of variations of Hodge structures, the intersection pairings do not match. According to Lemma 3.4, they differ by the order of the group *G*.

**Definition 3.9.** We define  $\eta_k^{\circ}$  as the trivializing section of  $(R^k f_* \Omega_{\mathcal{I}/U}^{n-1-k})_{\text{prim}}$ , deduced from  $\theta_k$  via the isomorphism (3.7) and by projecting to the Hodge bundle. It corresponds to the section  $\sigma_k$  above. We also define  $\eta_k = -(n+1)^{k+1} \psi^{k+1} \eta_k^{\circ}$ .

Remark 3.10. By construction, the section  $\eta_k$  vanishes at order k+1 at  $\psi=0$ .

Let now  $KS^{(q)}$  be the cup product with the Kodaira–Spencer class, induced by the Gauss–Manin connection:

(3.8) 
$$KS^{(q)} : T_U \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(R^q f_* \Omega_{\mathcal{Z}/U}^{n-1-q}, R^{q+1} f_* \Omega_{\mathcal{Z}/U}^{n-2-q}).$$

The algebraic theory of the Gauss–Manin connection [KO68] ensures that  $KS^{(q)}$  is already defined in the category of  $\mathbb{Q}$ -schemes. Somewhat abusively, we refer to  $KS^{(q)}$  as a Kodaira–Spencer morphism.

**Lemma 3.11.** The sections  $\eta_k^{\circ}$  satisfy the recurrence

(3.9) 
$$\operatorname{KS}^{(k)}\left(\frac{d}{d\psi}\right)\eta_{k}^{\circ} = (n+1)\eta_{k+1}^{\circ}.$$

Consequently,  $\eta_k^{\circ}$  is defined over  $\mathbb{Q}$  and

(3.10) 
$$KS^{(k)} \left( \psi \frac{d}{d\psi} \right) \eta_k = \eta_{k+1}.$$

*Proof.* The recurrence follows from (3.6), Proposition 3.7, the link between the Gauss–Manin connection  $\nabla$  and the Kodaira–Spencer maps  $\mathrm{KS}^{(q)}$ , and the definition of  $\eta_k^\circ$ . For the rationality statement, we first claim it for  $\eta_0^\circ$ . Indeed, the residue construction defining  $\theta_0$  makes sense in the algebraic category. The claim follows, since  $\eta_0^\circ$  is the image of  $\theta_0$  under (3.5), which is defined in the category of  $\mathbb{Q}$ -schemes. For the rest of sections, we apply the recurrence (3.9), together with the algebraicity property of  $\mathrm{KS}^{(k)}$  and the fact that  $d/d\psi$  is a section of  $T_{U/\mathbb{Q}}$ . Equation (3.10) follows from (3.9) by the definition  $\eta_k = -(n+1)^{k+1}\psi^{k+1}\eta_k^\circ$  and the  $\mathcal{O}_U$ -linearity of the Kodaira–Spencer maps.

An analogous argument can be carried out with the sections  $\theta_k$  and  $\theta'_k$ , thus proving the following statement.

**Lemma 3.12.** The isomorphism (3.7) already exists in the category of coherent sheaves on  $\mathbb{Q}$ -schemes.

#### 4. The degeneration of Hodge bundles of the Mirror Family

In the previous section we exhibited explicit trivializing sections of the middle degree Hodge bundles of the mirror family  $\mathcal{Z} \to U$ . The next goal is to extend these sections to the whole compactification  $\mathbb{P}^1$ . We also address the trivialization of the Hodge bundles outside the middle degree. For this purpose, we exploit the approach to degenerating Hodge structures via relative logarithmic de Rham cohomology.

Before embarking on our task, we recall some background from Steenbrink [Ste76, Ste77] and our previous work [EFiMM18a, Sec. 2 & Sec. 4]. Let  $f: X \to \mathbb{D}$  be a projective morphism of reduced analytic spaces, over the unit disc  $\mathbb{D}$ . We suppose that the fibers  $X_t$  with  $t \neq 0$  are smooth and connected. We consider the variation of Hodge structures defined by  $R^k f_* \mathbb{C}$  over the punctured disc  $\mathbb{D}^{\times}$ . Let T be its monodromy operator and  $\nabla$  the Gauss–Manin connection. Recall that T is a quasi-unipotent transformation of the cohomology of the general fiber. The flat vector bundle  $((R^k f_* \mathbb{C}) \otimes \mathcal{O}_{\mathbb{D}^\times}, \nabla)$  has a unique extension to a vector bundle with regular singular connection on  $\mathbb{D}$ , whose residue is an endomorphism with eigenvalues in  $[0,1) \cap \mathbb{Q}$ . This is the Deligne lower extension of  $R^k f_* \mathbb{C}$ , and we refer to it by  $\ell R^k f_* \mathbb{C}$ . It can be realized by the hypercohomology  $R^k f'\Omega^{ullet}_{X'/\mathbb{D}}$  (log) of the logarithmic de Rham complex of a normal crossing model  $f': X' \to \mathbb{D}$ . The Hodge filtration extends to a filtration by vector sub-bundles, with locally free graded quotients of the form  $R^{k-p} f' \Omega^p_{X'/\mathbb{D}}$  (log). If the monodromy operator is unipotent, then the fiber of  $R^k f'\Omega^{\bullet}_{X'/\mathbb{D}}(\log)$  at 0, together with the restricted Hodge filtration, can be identified with the cohomology of the generic fiber  $H^k_{\lim}$  with the limiting Hodge filtration  $F^{\bullet}_{\infty}$ . The identification depends on the choice of a holomorphic coordinate on  $\mathbb D$ . There is also the monodromy weight filtration  $W_{\bullet}$  on  $H_{\lim}^k$ , attached to the nilpotent operator N, the residue of the Gauss– Manin connection. The triple  $(H^k_{\lim}, F^{\bullet}_{\infty}, W_{\bullet})$  is called the limiting mixed Hodge structure. It is

isomorphic to Schmid's limiting mixed Hodge structure [Sch73]. In the general quasi-unipotent case, one first performs a semi-stable reduction and then constructs the limiting mixed Hodge structure.

Analogously, for a normal crossings degeneration  $f: X \to S$  between complex projective manifolds, there are algebraic counterparts of the logarithmic de Rham cohomology, Gauss-Manin connection, Hodge filtration, etc. This is compatible with the analytic theory after localizing to a holomorphic coordinate neighborhood of a given point  $p \in S$ . We will in particular speak of the limiting mixed Hodge structure at p, and simply write  $H_{lim}^k$  if there is no danger of confusion.

The foregoing discussion can be carried out in the polarized setting and for primitive cohomology. We will only consider polarizations induced by projective factorizations of our morphisms.

In the sequel, we specialize to the mirror family  $f: \mathcal{Z} \to \mathbb{P}^1$ . We fix the normal crossings model  $f': \mathcal{Z}' \to \mathbb{P}^1$  obtained by blowing-up the ordinary double points in the fibers  $Z_{\xi}$ , where  $\xi \in \mathbb{C}$  and  $\xi^{n+1} = 1$ . Given a polarization, say induced by a projective factorization of f', we study the limiting mixed Hodge structures on the middle primitive cohomology. To lighten notations, we write  $H_{\text{lim}}^{n-1}$  instead of  $H_{\text{prim, lim}}^{n-1}$ .

- 4.1. **Behaviour of**  $\eta_k$  **at the MUM point.** For the mirror family  $f: \mathcal{Z} \to \mathbb{P}^1$ , let  $\mathbb{D}_{\infty}$  be a holomorphic disc neighborhood at infinity, with parameter  $t = 1/\psi$ . To lighten notations, we still denote by  $f: \mathcal{Z} \to \mathbb{D}_{\infty}$  the restricted family. Also, following the previous conventions, we write  $H_{\lim}^{n-1}$ for the limiting mixed Hodge structure at infinity of the middle primitive cohomology.
- (1) The monodromy T of  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  at  $\infty$  is maximally unipotent. In par-Lemma 4.1.
  - ticular, the nilpotent operator N on  $H_{\lim}^{n-1}$  satisfies  $N^{n-1} \neq 0$ .

    (2) The graded pieces  $\operatorname{Gr}_k^W H_{\lim}^{n-1}$  are one-dimensional if k is even, and trivial otherwise. For all  $1 \le k \le n-1$ , N induces isomorphisms

$$\operatorname{Gr}_k^W N \colon \operatorname{Gr}_k^W H_{\lim}^{n-1} \xrightarrow{\sim} \operatorname{Gr}_{k-2}^W H_{\lim}^{n-1}.$$

(3) For all  $1 \le p \le n-1$ , N induces isomorphisms

$$\operatorname{Gr}_{F_{\infty}}^{p} N \colon \operatorname{Gr}_{F_{\infty}}^{p} H_{\lim}^{n-1} \xrightarrow{\sim} \operatorname{Gr}_{F_{\infty}}^{p-1} H_{\lim}^{n-1}.$$

*Proof.* In odd relative dimension, the maximally unipotent property for  $R^{n-1}f_*\mathbb{C}=(R^{n-1}h_*\mathbb{C})^G_{\mathrm{prim}}$ is [HSBT10, Cor. 1.7]. In even relative dimension, exactly the same argument as in *loc. cit.* yields the claim for  $(R^{n-1}h_*\mathbb{C})^G_{\text{prim}}$ . The property is inherited by  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  thanks to Proposition 3.7. In particular  $N^{n-1} \neq 0$ . This settles the first point. Because moreover  $N^{n-1}$  induces an isomorphism  $\operatorname{Gr}_{2(n-1)}^W H_{\lim}^{n-1} \xrightarrow{\sim} \operatorname{Gr}_0^W H_{\lim}^{n-1}$  we deduce that  $\operatorname{Gr}_0^W H_{\lim}^{n-1} \neq 0$ . Since  $H_{\lim}^{n-1}$  is ndimensional, the second item follows for dimension reasons. Finally, we use that  $\operatorname{Gr}_{F_{\infty}}^{p}H_{\lim}^{n-1}$  is one-dimensional by Lemma 3.6 and then necessarily  $\operatorname{Gr}_{F_{\infty}}^{p}H_{\lim}^{n-1}=\operatorname{Gr}_{F_{\infty}}^{p}\operatorname{Gr}_{2p}^{W}H_{\lim}^{n-1}=\operatorname{Gr}_{2p}^{W}H_{\lim}^{n-1}$ . Hence the second point implies the third one. Hence the second point implies the third one.

By the maximally unipotent monodromy and for dimension reasons, the *T*-invariant classes of the primitive cohomology of a general fiber span a rank one trivial sub-system of  $(R^{n-1}f_*\mathbb{C})_{prim}$ on  $\mathbb{D}_{\infty}^{\times}$ . We fix a basis  $\gamma'$  of this trivial system. It extends to a nowhere vanishing holomorphic section of the Deligne extension  $\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$ . The fiber at 0 is then a basis for  $W_0 = \ker N$ (this is not a general fact, but a special feature of the weight filtration under consideration). We still write  $\gamma'$  for this limit element. Similarly,  $(R^{n-1}f_*\mathbb{C})^\vee_{\text{prim}}$  has a rank one trivial sub-system,

spanned by the class of a T-invariant homological cycle  $\gamma$ . We may choose  $\gamma$  to correspond to  $\gamma'$  by Poincaré duality. Hence, for any  $\eta \in H^{n-1}(Z_t)$ ,  $t \in \mathbb{D}_{\infty}^{\times}$ , the period  $\langle \gamma, \eta \rangle$  equals the intersection pairing  $S(\gamma', \eta)$ . It is possible to explicitly construct an invariant cycle. Although we will need this in a moment, we postpone the discussion to §5.2, where a broader study of homological cycles is delivered.

**Lemma 4.2.** Let  $\eta$  be a holomorphic trivialization of  $f_*K_{\mathcal{Z}/\mathbb{D}_{\infty}}(\log)$ . Then the period  $\langle \gamma, \eta \rangle$  defines a holomorphic function on  $\mathbb{D}_{\infty}$ , non-vanishing at the origin.

*Proof.* The argument is well-known (see *e.g.* [Mor93, Prop.] and [Voi99, Lemma 3.10]), but we sketch it due to its relevance.

The pairing  $\langle \gamma, \eta \rangle = S(\gamma', \eta)$  is clearly a holomorphic function on  $\mathbb{D}_{\infty}^{\times}$ , since both  $\gamma'$  and  $\eta$  are holomorphic sections of  $(R^{n-1}f_*\mathbb{C}) \otimes \mathcal{O}_{\mathbb{D}_{\infty}^{\times}}$ . Moreover, they are both global sections of the Deligne extension. This ensures that  $|S(\gamma', \eta)|$  has at most a logarithmic singularity at 0. It follows that  $S(\gamma', \eta)$  is actually a holomorphic function.

For the non-vanishing property, we make use of the interplay between the intersection pairing seen on  $H_{\mathrm{lim}}^{n-1}$  and the monodromy weight filtration [Sch73, Lemma 6.4], together with Lemma 4.1. Let  $\eta' \in H_{\mathrm{lim}}^{n-1}$  be the fiber of  $\eta$  at 0. We need to show that  $S(\gamma', \eta') \neq 0$ . Suppose the contrary. Since  $\gamma'$  is a basis of  $W_0 = \ker N = \operatorname{Im} N^{n-1}$ , we have  $\eta' \in (\operatorname{Im} N^{n-1})^{\perp}$ . The intersection pairing is non-degenerate and satisfies S(Nx, y) + S(x, Ny) = 0. Therefore, we find that  $\eta' \in (\operatorname{Im} N^{n-1})^{\perp} = \ker N^{n-1} = W_{2n-3}$ . But  $\eta'$  is a basis of  $F^{n-1}H_{\mathrm{lim}}^{n-1} = F^{n-1}\operatorname{Gr}_{2n-2}^W H_{\mathrm{lim}}^{n-1}$ , and therefore  $\eta' \not\in W_{2n-3}$ . We thus have reached a contradiction.

Before the next theorem, we consider the logarithmic extension of the Kodaira–Spencer maps (3.8): if D is the divisor  $[\infty] + \sum_{\xi^{n+1}=1} [\xi]$ , then

They preserve the primitive components.

**Theorem 4.3.** The section  $\eta_k$  is a holomorphic trivialization of  $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_{\infty}}^{n-1-k}(\log)_{\text{prim}}$ .

*Proof.* First of all, we prove that  $\eta_0$  is a meromorphic section of  $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ . Indeed,  $\eta_0$  is an algebraic section of  $f_*K_{\mathcal{Z}/U}$  (see Lemma 3.11), hence a rational section of  $f_*K_{\mathcal{Z}'/\mathbb{P}^1}(\log)$  and thus a meromorphic section of  $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ .

Second, we establish the claim of the theorem for  $\eta_0$ . By Lemma 4.2, we need to show that the holomorphic function  $\langle \gamma, \eta_0 \rangle$  on  $\mathbb{D}_{\infty}^{\times}$  extends holomorphically to  $\mathbb{D}_{\infty}$ , and does not vanish at the origin. This property can be checked by a standard explicit computation reproduced below (5.6).

Finally, for the sections  $\eta_k$ , we use the recurrence (3.10) and the logarithmic extension of the Kodaira–Spencer maps (4.1). It follows that the sections  $\eta_k$  are global sections of the sheaves  $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k} (\log)_{\text{prim}}$ . Let us denote by  $\eta'_k$  the fiber at 0 of the sections  $\eta_k$ . Specializing (3.10) at 0, we find  $(\operatorname{Gr}_{F_\infty}^{n-1-k} N) \eta'_k = \eta'_{k+1}$ . By Lemma 4.1 (3) and because  $\eta'_0 \neq 0$ , we see that  $\eta'_k \neq 0$  for all k. This concludes the proof.

4.2. **Behaviour of**  $\eta_k$  **at the ODP points.** Recall the normal crossings model  $f' \colon \mathcal{Z}' \to \mathbb{P}^1$ . We restrict it to a disc neighborhood  $\mathbb{D}_{\xi}$  of some  $\xi \in \mu_{n+1}$ . Concretely, we fix the coordinate  $t = \psi - \xi$ . We write  $f' \colon \mathcal{Z}' \to \mathbb{D}_{\xi}$  for the restricted family. We now deal with the limiting mixed Hodge structure  $H_{\lim}^{n-1}$  at  $\xi$  of the middle primitive cohomology. Since the monodromy around  $\xi$  is not

unipotent in general, the construction of  $H_{\text{lim}}^{n-1}$  requires a preliminary semi-stable reduction. This can be achieved as follows:

Hence  $\tilde{f}: \tilde{\mathcal{Z}} \to \mathbb{D}_{\xi}$  is the normalized base change of f' by  $\rho$ . An explicit computation in local coordinates shows it is indeed semi-stable. The special fiber  $\tilde{f}^{-1}(0)$  consists of two components intersecting transversally. One is the strict transform  $\widetilde{Z}$  of  $Z_{\xi}$ . We denote by E the other component. Then E is a non-singular quadric of dimension n-1, and  $\widetilde{Z} \cap E$  is a non-singular quadric of dimension n-2. In terms of this data, the monodromy weight filtration is computed as follows, cf. [Ste77, Ex. 2.15].

**Lemma 4.4.** The graded pieces of the weight filtration on  $H_{lim}^{n-1}$  are given by:

• if n-1 is odd, then

$$\operatorname{Gr}_{k}^{W}H_{\lim}^{n-1} = \begin{cases} \mathbb{Q}\left(-\frac{n-2}{2}\right), & \text{if } k = n-2, \\ H^{n-1}(\widetilde{Z}), & \text{if } k = n-1, \\ \mathbb{Q}\left(-\frac{n}{2}\right), & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

• if n-1 is even, then

$$\operatorname{Gr}_{k}^{W}H_{\lim}^{n-1} = \begin{cases} H\left(H^{n-3}(\widetilde{Z}\cap E)(-1) \to H^{n-1}(\widetilde{Z}) \oplus H^{n-1}(E) \to H^{n-1}(\widetilde{Z}\cap E)\right), & \text{if } k=n-1, \\ 0, & \text{if } k\neq n-1. \end{cases}$$

$$Hence, H_{\lim}^{n-1} \text{ is a pure Hodge structure of weight } n-1.$$

We will need the comparison of the middle degree Hodge bundles between before and after semi-stable reduction. We follow [EFiMM18a, Sec. 2 & Prop. 3.10]. There are natural morphisms

$$(4.3) \varphi^{p,q} : \rho^* R^q f'_* \Omega^p_{\mathcal{Z}'/\mathbb{D}_{\xi}} (\log)_{\text{prim}} \hookrightarrow R^q \widetilde{f}_* \Omega^p_{\widetilde{\mathcal{Z}}/\mathbb{D}_{\xi}} (\log)_{\text{prim}}.$$

**Lemma 4.5.** Suppose that p + q = n - 1. Let  $Q^{p,q}$  be the cokernel of  $\varphi^{p,q}$  in (4.3).

- If  $p \neq q$ , then  $Q^{p,q} = 0$ . If  $p = q = \frac{n-1}{2}$ , then  $Q^{p,p} = \mathcal{O}_{\mathbb{D}_{\xi},0} / u\mathcal{O}_{\mathbb{D}_{\xi},0}$ .

*Proof.* The results in [EFiMM18a, Sec. 2 & Prop. 3.10] are explicitly stated for the whole Hodge bundles. For their primitive components, see however Remark 2.7 (iii) in loc. cit., or notice an easy compatibility with the primitive decomposition.

The last fact we need is the computation of the Yukawa coupling. A repeated application of the Kodaira-Spencer maps gives a morphism

$$(4.4) Y: \operatorname{Sym}^{n-1} T_U \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(f_* K_{\mathcal{Z}/U}, R^{n-1} f_* \mathcal{O}_{\mathcal{Z}}) \simeq (f_* K_{\mathcal{Z}/U})^{\otimes -2}.$$

Using the section  $\psi d/d\psi$  of  $T_U$  and the section  $\eta_0$  of  $f_*K_{\mathcal{Z}/U}$ , we obtain a holomorphic function on U, denoted  $Y(\psi)$ . Working with  $(R^{n-1}h_*\Omega^{\bullet}_{\mathcal{Z}/U})^G$  instead, one similarly defines a function

 $\widetilde{Y}(\psi)$ . Via the isomorphism of Proposition 3.7, the functions  $\widetilde{Y}(\psi)$  and  $Y(\psi)$  can be compared. The only subtle point to bear in mind is the use of Serre duality in the definition of the Yukawa coupling. For Hodge bundles of complementary bi-degree, Serre duality is induced by the cupproduct and the trace morphism. Hence, an application of Lemma 3.4 shows that  $Y(\psi)$  and  $\widetilde{Y}(\psi)$  are equal up to the order of G. With this understood, we can invoke the computation of the Yukawa coupling in [BvS95, Cor. 4.5.6 & Ex. 4.5.7], and conclude

(4.5) 
$$Y(\psi) = c \frac{\psi^{n+1}}{1 - \psi^{n+1}}$$

for some constant  $c \neq 0$ . With notations as in *loc. cit.*, their factor  $\lambda z$  is  $1/\psi^{n+1}$ , thus explaining the formal discrepancy of both formulas.

We are now fully equipped for the proof of:

**Theorem 4.6.** The sections  $\eta_k$  extend to rational sections of the logarithmic Hodge bundles  $R^k f'_* \Omega_{\mathcal{Z}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$ . Furthermore, if  $\operatorname{ord}_{\xi} \eta_k$  is the order of zero or pole of  $\eta_k$  at  $\xi$ , as a rational section of  $R^k f'_* \Omega_{\mathcal{Z}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$ , then:

- *if* n-1 *is odd, then*  $\operatorname{ord}_{\xi} \eta_k = 0$  *for*  $k \le n/2 1$  *and*  $\operatorname{ord}_{\xi} \eta_k = -1$  *otherwise.*
- if n-1 is even, then  $\operatorname{ord}_{\xi} \eta_k = 0$  for  $k \leq \frac{n-3}{2}$  and  $\operatorname{ord}_{\xi} \eta_k = -1$  otherwise.

*Proof.* Throughout the proof, we write  $\mathscr{X}$ ,  $\mathscr{Y}$  and  $\mathscr{Z}$  for the respective total spaces over  $\mathbb{A}^1$ . We begin by showing that  $\eta_0$  extends to a global section of  $f'_*K_{\mathscr{Z}'/\mathbb{A}^1}(\log)$ , non-vanishing at  $\xi$ . Since the singular fibers of  $\mathscr{Z} \to \mathbb{A}^1$  present only ordinary double points, there is an equality

$$f_*K_{\mathcal{T}/\mathbb{A}^1} = f'_*K_{\mathcal{T}'/\mathbb{A}^1}(\log).$$

This can be seen as the coincidence of the upper and lower extensions of  $f_*K_{\mathcal{Z}/U}$  to  $\mathbb{A}^1$  (apply [EFiMM18a, Cor. 2.8 & Prop. 2.10] and the Picard–Lefschetz formula for the monodromy). Since  $\mathscr{Y}$  has rational singularities (cf. Lemma 3.1), the natural morphism  $g_*K_{\mathscr{Y}/\mathbb{A}^1} \to f_*K_{\mathcal{Z}/\mathbb{A}^1}$  is an isomorphism. Also  $g_*K_{\mathscr{Y}/\mathbb{A}^1} = (h_*K_{\mathscr{X}/\mathbb{A}^1})^G$ . Indeed, let  $\mathscr{X}^\circ$  be the complement of the fixed point locus of G in  $\mathscr{X}$  and similarly for  $\mathscr{Y}^\circ$ , so that  $\mathscr{Y} \setminus \mathscr{Y}^\circ$  has codimension  $\geq$  2. Then, because  $\mathscr{Y}$  is normal Gorenstein and  $\mathscr{Y}^\circ = \mathscr{X}^\circ/G$  is an étale quotient, and  $\mathscr{X}$  is non-singular, we find

$$g_*K_{\mathscr{Y}/\mathbb{A}^1} = g_*K_{\mathscr{Y}^\circ/\mathbb{A}^1} = (h_*K_{\mathscr{Y}^\circ/\mathbb{A}^1})^G = (h_*K_{\mathscr{Y}/\mathbb{A}^1})^G.$$

By construction of  $\eta_0$  (cf. Definition 3.9), it is enough to prove that  $\theta_0$  defines a trivialization of  $h_*K_{\mathcal{X}/\mathbb{A}^1}$  around  $\xi$ . Denote by  $\mathcal{X}^*$  the complement in  $\mathcal{X}$  of the ordinary double points, so that  $\mathcal{X} \setminus \mathcal{X}^*$  has codimension  $\geq 2$ . Because  $\mathcal{X}$  is non-singular, we have  $h_*K_{\mathcal{X}/\mathbb{A}^1} = h_*K_{\mathcal{X}^*/\mathbb{A}^1}$ . Now, the expression (3.1) for  $\theta_0$  defines a relative holomorphic volume form on the whole  $\mathcal{X}^*$ , and hence a trivialization of  $h_*K_{\mathcal{X}^*/\mathbb{A}^1}$  as desired.

That the sections  $\eta_k$  define rational sections of the sheaves  $R^k f'_* \Omega_{\mathcal{I}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$  follows from the corresponding property for  $\eta_0$ , plus the recurrence (3.10) and the existence of the logarithmic extension of the Kodaira–Spencer maps (4.1). From the same recurrence, we reduce the computation of  $\operatorname{ord}_{\xi} \eta_k$  to the computation of the orders at  $\xi$  of the rational morphisms  $\operatorname{KS}^{(j)}(\psi d/d\psi)$ , with respect to the logarithmic extension of the Hodge bundles:

$$\operatorname{ord}_{\xi} \eta_{k} = \operatorname{ord}_{\xi} \eta_{0} + \sum_{j=0}^{k-1} \operatorname{ord}_{\xi} \operatorname{KS}^{(j)} \left( \psi \frac{d}{d\psi} \right) = \sum_{j=0}^{k-1} \operatorname{ord}_{\xi} \operatorname{KS}^{(j)} \left( \psi \frac{d}{d\psi} \right).$$

Let us define  $M^{(j)} = \operatorname{ord}_{\xi} \operatorname{KS}^{(j)} \left( \psi \frac{d}{d\psi} \right)$ . Because  $\eta_0$  trivializes  $f_* K_{\mathcal{Z}/\mathbb{A}^1}$  at  $\xi$ , formula (4.5) shows that

(4.6) 
$$\sum_{j=0}^{n-2} M^{(j)} = \operatorname{ord}_{\xi} Y(\psi) = -1.$$

We argue that all but one of the  $M^{(j)}$  are in fact zero. For this, we relate  $M^{(j)}$  to the action of the nilpotent operator N on the limiting mixed Hodge structure at  $\xi$ . Recall we defined the coordinate  $t = \psi - \xi$  on a disc neighborhood  $\mathbb{D}_{\xi}$  of  $\xi$ . The first observation is

$$\operatorname{ord}_{t=0} \operatorname{KS}^{(j)} \left( t \frac{d}{dt} \right) = \operatorname{ord}_{\xi} \operatorname{KS}^{(j)} \left( (\psi - \xi) \frac{d}{d\psi} \right) = M^{(j)} + 1.$$

We now need to distinguish two cases, depending on the parity of n-1.

*Odd case*: If n-1 is odd, then the monodromy is unipotent and the fiber of KS<sup>(p)</sup>(td/dt) at t=0 is already  $\operatorname{Gr}_{F_{\infty}}^{p}N$ :  $\operatorname{Gr}_{F_{\infty}}^{p}H_{\lim}^{n-1} \to \operatorname{Gr}_{F_{\infty}}^{p-1}H_{\lim}^{n-1}$ . From Lemma 4.4, we deduce that unless p=n/2,  $\operatorname{Gr}_{F_{\infty}}^{p}N=0$  so that  $\operatorname{ord}_{t=0}\operatorname{KS}^{(p)}(td/dt)>0$  and hence  $M^{(p)}\geq 0$ . By (4.6) we necessarily have  $M^{(n/2)}=-1$  and the other  $M^{(j)}=0$ .

*Even case*: If n-1 is even, the nilpotent operator N is in fact trivial, but the monodromy is no longer unipotent. The construction of the limiting mixed Hodge structure thus involves a semi-stable reduction. Choose a square root u of t as in (4.2). Then since  $u\frac{d}{du} = 2t\frac{d}{dt}$  and  $\operatorname{ord}_{u=0} = 2\operatorname{ord}_{t=0}$  we get

(4.7) 
$$\operatorname{ord}_{u=0} \varphi^{p,q} + \operatorname{ord}_{u=0} \operatorname{KS}^{(q)} \left( u \frac{d}{du} \right) = \operatorname{ord}_{u=0} (\varphi^{p-1,q+1}) + 2 \operatorname{ord}_{t=0} \operatorname{KS}^{(q)} \left( t \frac{d}{dt} \right).$$

By Lemma 4.5,  $\operatorname{ord}_{u=0}(\varphi^{p,q}) = 0$  except for the case (p,q) = ((n-1)/2, (n-1)/2), where in fact  $\operatorname{ord}_{u=0}(\varphi^{p,q}) = 1$ . From (4.7) we then conclude that

(4.8) 
$$\operatorname{ord}_{u=0} KS^{((n-3)/2)} \left( u \frac{d}{du} \right) = 1 + 2 \operatorname{ord}_{t=0} KS^{((n-3)/2)} \left( t \frac{d}{dt} \right)$$

(4.9) 
$$1 + \operatorname{ord}_{u=0} KS^{((n-1)/2)} \left( u \frac{d}{du} \right) = 2 \operatorname{ord}_{t=0} KS^{((n-1)/2)} \left( t \frac{d}{dt} \right).$$

In both cases (4.8)–(4.9) the order of vanishing of Kodaira–Spencer along the vector field  $u \frac{d}{du}$  is strictly positive, since the restriction to 0 is the nilpotent operator N = 0. It follows that

$$\operatorname{ord}_{t=0} \operatorname{KS}^{((n-3)/2)} \left( t \frac{d}{dt} \right) \ge 0, \quad i.e. \quad M^{((n-3)/2)} \ge -1,$$

and

$$\operatorname{ord}_{t=0} \operatorname{KS}^{((n-1)/2)} \left( t \frac{d}{dt} \right) \ge 1, \quad i.e. \quad M^{((n-1)/2)} \ge 0.$$

Since all other  $M^{(j)} \ge 0$  as in the odd case, we conclude from (4.6) that all these inequalities are in fact equalities.

4.3. **Triviality of the Hodge bundles outside the middle degree.** Recall the normal crossings model  $f' \colon \mathcal{Z}' \to \mathbb{P}^1$ , obtained by blowing-up the ordinary double points in  $\mathcal{Z}$ . Notice that f' is actually defined over  $\mathbb{Q}$ , and hence so are the corresponding logarithmic Hodge bundles. By Lemma 3.6 we have  $R^d f'_* \Omega^{\bullet}_{\mathcal{Z}'/\mathbb{P}^1}(\log) = 0$  for d odd, not equal to n-1, while if  $d = 2p \neq n-1$ ,  $R^d f'_* \Omega^{\bullet}_{\mathcal{Z}'/\mathbb{P}^1}(\log) = R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ .

**Lemma 4.7.** For  $2p \neq n-1$ , the following hold:

- (1) the local system  $R^{2p} f_* \mathbb{Q}$  on  $U(\mathbb{C}) = \mathbb{C} \setminus \mu_{n+1}$  is trivial.
- (2) the Hodge bundle  $R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$  is trivial in the category of coherent sheaves on  $\mathbb{Q}$ -schemes.

*Proof.* We first prove that the local system  $R^{2p}f_*\mathbb{Q}$  is trivial. Take a base point  $b \in U(\mathbb{C})$ , and let  $\rho \colon \pi_1(U(\mathbb{C}),b) \to \mathsf{GL}(H^{2p}(Z_b,\mathbb{Q}))$  be the monodromy representation determining the local system. The fundamental group  $\pi_1(U(\mathbb{C}),b)$  is generated by loops  $\gamma_\xi$  circling around  $\xi \in \mu_{n+1}$ , and a loop  $\gamma_\infty$  circling around  $\infty$ , with a relation  $\prod_\xi \gamma_\xi = \gamma_\infty$ . Because the singularities of  $\mathcal{Z} \to \mathbb{P}^1$  at the points  $\xi$  are ordinary double points, and  $2p \neq n-1$ , the local monodromies  $\rho(\gamma_\xi)$  are trivial. Therefore  $\rho(\gamma_\infty)$  is trivial as well, and so is  $\rho$ .

Now, the first claim implies the triviality of  $R^pf'_*\Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log) = R^{2p}f'_*\Omega^\bullet_{\mathcal{Z}'/\mathbb{P}^1}(\log)$  over  $\mathbb{C}$ , since the latter is the Deligne extension of  $R^{2p}f_*\mathbb{C}$ . This already implies the second claim. Indeed, let E be a vector bundle over  $\mathbb{P}^1_\mathbb{Q}$ , which is trivial after base change to  $\mathbb{C}$ . Then the natural morphism  $H^0(\mathbb{P}^1_\mathbb{Q},E)\otimes \mathcal{O}_{\mathbb{P}^1_\mathbb{Q}}\to E$  is necessarily an isomorphism, since it is an isomorphism after a flat base change.

5. The BCOV invariant of the mirror family

5.1. The Kronecker limit formula for the mirror family. For the mirror family  $f: \mathcal{Z} \to U$ , we proceed to prove an expression for the BCOV invariant  $\tau_{BCOV}(Z_{\psi})$  in terms of the  $L^2$  norms of the sections  $\eta_k$  (cf. Definition 3.9). The strategy follows the same lines as for families of Calabi-Yau hypersurfaces §2.4.

We fix a polarization and a projective factorization of f, defined over  $\mathbb Q$ . We denote by L the corresponding algebraic Lefschetz operator, that is the cup-product against the algebraic cycle class of a hyperplane section. We will abusively confound L with the algebraic cycle class of a hyperplane section. With this choice of L, the primitive decomposition of the Hodge bundles  $R^p f_* \Omega^q_{\mathcal{I}/U}$  holds over  $\mathbb Q$ . Let h be a Kähler metric and  $\omega$  the Kähler form normalized as in (2.1), and assume that the fiberwise cohomology class is in the topological hyperplane class. Hence, under the correspondence between algebraic and topological cycle classes, L is sent to  $(2\pi i)[\omega] \in R^2 f_* \mathbb Q(1)$ .

Below, all the  $L^2$  norms are computed with respect to  $\omega$  as in (2.2).

**Theorem 5.1.** There exists a real positive constant  $C \in \pi^c \overline{\mathbb{Q}}^{\times}$  such that

$$\tau_{BCOV}(Z_{\psi}) = C \left| \frac{(\psi^{n+1})^a}{(1 - \psi^{n+1})^b} \right|^2 \frac{\|\eta_0\|_{L^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\eta_k\|_{L^2}^{2(n-1-k)}\right)^{(-1)^{n-1}}}$$

where  $\chi = \chi(Z_{\psi})$  and

$$a = (-1)^{n-1} \frac{n(n-1)}{6} - \frac{\chi}{12(n+1)},$$

$$b = (-1)^{n-1} \frac{n(3n-5)}{24}$$

$$c = \frac{1}{2} \sum_{k} (-1)^{k+1} k^2 b_k.$$

*Proof.* We apply the version of the arithmetic Riemann–Roch theorem formulated in Theorem 2.3, to the family  $f: \mathcal{Z} \to U$  as being defined over  $\mathbb{Q}$ . We need to specify the section  $\eta$  and the sections  $\eta_{p,q}$  in equation (2.10). The section  $\eta$  is chosen to be  $\eta_0$  as defined in Definition 3.9. We next describe our choices of  $\eta_{p,q}$ :

- If  $p + q \neq n 1$  and  $p \neq q$ , then the corresponding Hodge bundle vanishes by Lemma 3.6, and thus gives no contribution.
- For  $2p \neq n-1$ , Lemma 4.7 guarantees that  $\det R^p f'_* \Omega^p_{\mathcal{I}'/\mathbb{P}^1}(\log)$  is trivial, in the category of  $\mathbb{Q}$ -schemes. We choose  $\eta_{p,p}$  to be any trivialization defined over  $\mathbb{Q}$ , and then restrict it to U. Notice that the  $L^2$  norm  $\|\eta_{p,p}\|_{L^2}$  is constant.
- For p+q=n-1 and  $p\neq q$ , the (p,q) Hodge bundle is primitive and has rank one. Then we take  $\eta_{p,q}=\eta_q$  in Definition 3.9. By Lemma 3.11,  $\eta_q$  is defined over  $\mathbb Q$ .
- For p + q = n 1 and p = q, which can only occur when n 1 is even, the (p, q) Hodge bundle is no longer primitive. We first employ the algebraic primitive decomposition:

(5.1) 
$$\det R^{p} f_{*} \Omega_{\mathcal{Z}/U}^{p} = \det (R^{p} f_{*} \Omega_{\mathcal{Z}/U}^{p})_{\text{prim}} \otimes \det L R^{p-1} f_{*} \Omega_{\mathcal{Z}/U}^{p-1}$$
$$\simeq \det (R^{p} f_{*} \Omega_{\mathcal{Z}/U}^{p})_{\text{prim}} \otimes \det R^{p-1} f_{*} \Omega_{\mathcal{Z}/U}^{p-1}.$$

We define  $\eta_{\frac{n-1}{2},\frac{n-1}{2}}$  as the element corresponding to  $\eta_{\frac{n-1}{2}}\otimes\eta_{\frac{n-3}{2},\frac{n-3}{2}}$  under this isomorphism. Again, this element is defined over  $\mathbb{Q}$ .

To establish the theorem we need to specify the element  $\Delta \in \mathbb{Q}(\psi)^{\times} \otimes \mathbb{Q}$  in (2.10) (formal rational power of a rational function), which satisfies:

(5.2) 
$$\log \tau_{BCOV} = \log |\Delta|^2 + \frac{\chi}{12} \log \|\eta\|_{L^2}^2 - \sum_{p,q} (-1)^{p+q} p \log \|\eta_{p,q}\|_{L^2}^2 + \log C_{\sigma}.$$

We will determine  $\Delta$  up to an algebraic number. To this end, it suffices to know its divisor. Unless  $\psi=0$  or  $\psi=\xi$  where  $\xi^{n+1}=1$ ,  $\Delta$  has no zeroes no poles by construction, since the sections  $\eta_{p,q}$  are holomorphic and non-vanishing, and  $\log \tau_{BCOV}$  is smooth. Hence we are lead to consider the logarithmic behaviour of the right hand side of (5.2) at these points. Since for  $2p\neq n-1$  the sections  $\eta_{p,p}$  have constant  $L^2$  norm, we only need to examine the functions  $\log \|\eta_{p,q}\|_{L^2}$  with p+q=n-1.

Behaviour at  $\psi = 0$ . This corresponds to a smooth fiber of  $f: \mathcal{Z} \to U$ . Hence  $\log \tau_{BCOV}$  is smooth at  $\psi = 0$ , as are the  $L^2$  metrics. However, the sections  $\eta_{p,q}$  with p+q=n-1 admit zeros at  $\psi = 0$  (see Remark 3.10), with  $\operatorname{ord}_0 \eta_{p,q} = q+1=n-p$ . This means that a in the theorem is given by

$$(n+1)a = (-1)^{n-1} \sum_{p=0}^{n-1} p(n-p) - \frac{\chi}{12} = (-1)^{n-1} \frac{(n-1)n(n+1)}{6} - \frac{\chi}{12}.$$

Behaviour at  $\psi = \xi \in \mu_{n+1}$ . This corresponds to a singular fiber of  $f: \mathcal{Z} \to \mathbb{P}^1$ , which has a unique ordinary double point. By Theorem 4.6 we control  $\operatorname{ord}_{\xi} \eta_k$  according to the parity of n-1. Here we encounter the additional problem that the  $L^2$  norms might have contributions from the semi-simple part of the monodromy  $T_s$ . More precisely, consider the local parameter  $t = \psi - \xi$  around  $\xi$ , and write  $\eta_{p,q} = t^{b_{p,q}} \sigma_{p,q}$  where  $\sigma_{p,q}$  trivializes  $\det R^q f_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ . Then by [EFiMM18a, Thm. C], we have

$$\log \|\eta_{p,q}\|_{L^2}^2 = (b_{p,q} + \alpha_{p,q}) \log |t|^2 + o(\log |t|^2)$$

with

$$\alpha_{p,q} = -\frac{1}{2\pi i} \operatorname{tr}\left(\ell \log T_s \mid \operatorname{Gr}_{F_{\infty}}^p H_{\lim}^{n-1}\right) \in \mathbb{Q}.$$

Here  $^{\ell}$  log refers to the lower branch of the logarithm, *i.e.* with argument in  $2\pi(-1,0]$ . In the case at hand, this exponent can be determined from Lemma 4.5. Let us combine all this information:

Odd case: If n-1 is odd, according to Theorem 4.6 , if  $k \leq \frac{n}{2}-1$ ,  $\operatorname{ord}_{\xi} \eta_k = 0$  and  $\operatorname{ord}_{\xi} \eta_k = -1$  otherwise. In this case the monodromy is unipotent, so that  $\alpha_{p,q} = 0$  for all p+q=n-1. Moreover, by [EFiMM18a, Thm. B], we have that  $\log \tau_{BCOV} = \frac{n}{24} \log |t|^2 + o(\log |t|^2)$ . Putting all these contributions together we find that

$$b = \frac{n}{24} + (-1)^{n-1} \sum_{k=n/2}^{n-1} (n-1-k) \cdot (-1) = \frac{n(3n-5)}{24}.$$

Even case: If n-1 is even, according to Theorem 4.6 , if  $k \leq \frac{n-3}{2}$ ,  $\operatorname{ord}_{\xi} \eta_k = 0$  and  $\operatorname{ord}_{\xi} \eta_k = -1$  otherwise. Also, unless p = q = (n-1)/2,  $\alpha_{p,q} = 0$ . In the remaining case p = q = (n-1)/2, Lemma 4.5 implies that  $\alpha_{p,p} = 1/2$ . Finally, from [EFiMM18a, Thm. B], we have that  $\log \tau_{BCOV} = \frac{3-n}{24} \log |t|^2 + o(\log |t|^2)$ . Putting all these contributions together we find that

$$b = \frac{3-n}{24} + (-1)^{n-1} \left( (n-1)/2(-1+1/2) + \sum_{k=(n+1)/2}^{n-1} (n-1-k) \cdot (-1) \right) = -\frac{n(3n-5)}{24}.$$

To complete the proof of the theorem, we still need to tackle the constant C. There are two sources that contribute: i) for  $2p \neq n-1$ , the  $L^2$  norms  $\|\eta_{p,p}\|_{L^2}$  are constant and ii) if n-1=2p, then  $\eta_{p,p}$  was taken to correspond to  $\eta_p \otimes \eta_{p-1,p-1}$  through (5.1), so that there might be extra contributions from L and from  $\|\eta_{p-1,p-1}\|_{L^2}$ .

First for  $2p \neq n-1$ . Let  $\psi \in \mathbb{Q}$ , so that we have the period isomorphism

$$H^{2p}(Z_{\Psi},\Omega_{Z_{w}/\mathbb{Q}}^{\bullet})\otimes_{\mathbb{Q}}\mathbb{C}\stackrel{\sim}{\longrightarrow} H^{2p}(Z_{\Psi},\mathbb{Q})\otimes\mathbb{C}.$$

Taking rational bases on both sides, the determinant can be defined in  $\mathbb{C}^{\times}/\mathbb{Q}^{\times}$ . It equals  $(2\pi)^{pb_{2p}}$ . Since  $\|\eta_{p,p}\|_{L^2}$  is constant, it can be evaluated at any  $\psi \in \mathbb{Q}$ . We find

$$\|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} (2\pi)^{2pb_{2p}} \operatorname{vol}_{L^2}(H^{2p}(Z_{\psi},\mathbb{Z}),\omega),$$

where  $\sim_{\mathbb{Q}^\times}$  means equality up to a non-zero rational number. Now recall from (2.13) that with the Arakelov theoretic normalization of the Kähler form, and under the integrality assumption on its cohomology class, we have  $\operatorname{vol}_{L^2}(H^{2p}(Z_\psi,\mathbb{Z}),\omega) \sim_{\mathbb{Q}^\times} (2\pi)^{-2pb_{2p}}$ . All in all, we arrive at the pleasant

(5.3) 
$$\|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} 1.$$

If 2p = n - 1,  $\eta_{p,p}$  corresponds to  $\eta_p \otimes \eta_{p-1,p-1}$  through (5.1). We bring together several facts. The first one is that the Lefschetz decomposition is orthogonal for the  $L^2$  metrics, regardless of the normalization of the Kähler forms. The second one is that the algebraic cycle class of L corresponds to  $(2\pi i)[\omega]$  in analytic de Rham cohomology. The last fact is that the operator  $[2\pi\omega] \wedge \cdot$  is an isometry up to a rational constant, since  $2\pi\omega$  is the Hodge theoretic Kähler form (see for instance [Huy05, Prop. 1.2.31]). All these remarks together lead to

$$\|\eta_{p,p}\|_{L^{2}}^{2} \sim_{\mathbb{Q}^{\times}} \|\eta_{p}\|_{L^{2}}^{2} \|\eta_{p-1,p-1}\|_{L^{2}}^{2}.$$

Appying (5.3) to  $\|\eta_{p-1,p-1}\|_{L^2}$ , we find again

$$\|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} 1.$$

Now plug (5.3)–(5.4) into (5.2), introduce as well the value of  $C_{\sigma}$  (cf. Theorem 2.3) and recall that  $\Delta$  was determined only up to algebraic number. We conclude that C has the asserted shape.  $\square$ 

**Corollary 5.2.** As  $\psi \to \infty$ ,  $\log \tau_{BCOV}(Z_{\psi})$  behaves as

(5.5) 
$$\log \tau_{BCOV}(Z_{\psi}) = \kappa_{\infty} \log |\psi|^{-2} + \rho_{\infty} \log \log |\psi|^{-2} + continuous,$$

where

$$\kappa_{\infty} = (-1)^{n} \frac{n+1}{12} \left( \frac{(n-1)(n+2)}{2} + \frac{1-(-n)^{n+1}}{(n+1)^{2}} \right),$$

$$\varrho_{\infty} = (-1)^{n-1} \frac{(n-1)(n+1)}{12} \left( \frac{(-n)^{n+1}-1}{(n+1)^{2}} - 2n + 1 \right).$$

*Proof.* The general shape (5.5) was proven in [EFiMM18a, Prop. 6.8]. The precise value of  $\kappa_{\infty}$  is (n+1)(b-a) entirely due to the term  $\left|\frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b}\right|$  in Theorem 5.1. Indeed, by Theorem 4.3 the sections  $\eta_k$  trivialize  $R^k f_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^{n-1-k}$  (log) at infinity, and moreover the monodromy is unipotent there (Lemma 4.1). This entails that the functions  $\log \|\eta_k\|_{L^2}^2$  are  $O(\log\log|\psi|^{-2})$  at infinity, and hence do not contribute to  $\kappa_{\infty}$ . For the subdominant term, the expression of [EFiMM18a, Prop. 6.8] can be explicitly evaluated for the mirror family, thanks to the complete understanding of the limiting Hodge structure at infinity (again Lemma 4.1), and the known value of  $\chi$  (Lemma 3.6).

#### 5.2. Canonical trivializations of the Hodge bundles at the MUM point.

The Picard–Fuchs equation of the mirror. For the mirror family  $f: \mathcal{Z} \to U$ , we review classical facts on the Picard–Fuchs equation of the local system of middle degree cohomologies. The discussion serves as the basis for the construction of canonical trivializing sections of the middle degree Hodge bundles, close to the MUM point, which differ from the  $\eta_k$  by some periods.

The starting point is the construction of an invariant (n-1)-homological cycle at infinity for the mirror family  $f\colon \mathcal{Z}\to \mathbb{P}^1$ . Recall the Dwork pencil  $h\colon \mathcal{X}\to \mathbb{P}^1$ , which comes with a natural embedding in  $\mathbb{P}^n\times \mathbb{P}^1$ . We obtain a "physical" n-cycle  $\Gamma$  in  $\mathbb{P}^n$  as follows: we place ourselves in the affine piece  $x_0\neq 0$  and define  $\Gamma$  by the condition  $|x_i/x_0|=1$  for all i. If  $\psi\in\mathbb{C}$  and  $|\psi|^{-1}$  is small, then the fiber  $X_\psi$  does not encounter  $\Gamma$ . Therefore,  $\Gamma$  induces a constant family of cycles in  $H_n(\mathbb{P}^n\setminus X_\psi,\mathbb{Z})$ . Notice that these are clearly G-invariant cycles. Under the tube isomorphism  $H_n(\mathbb{P}^n\setminus X_\psi,\mathbb{Z})\simeq H_{n-1}(X_\psi,\mathbb{Z})$ , which is G-equivariant, we find a T-invariant cycle  $\widetilde{\gamma}_0\in H_{n-1}(X_\psi,\mathbb{Z})^G$ . Finally, through  $H_{n-1}(X_\psi,\mathbb{Q})^G\hookrightarrow H_{n-1}(Z_\psi,\mathbb{Q})$  (cf. §3.2),  $|G|\cdot\widetilde{\gamma}_0$  maps

to a T-invariant cycle on  $Z_{\psi}$ , denoted  $\gamma_0$ . The convenience of multiplication by |G| will be clear in a moment.

The period integral  $I_0(\psi) := \int_{\gamma_0} \eta_0$  can be written as a convergent power series in  $\psi^{-1}$ . Indeed, taking into account the relationship between the cup-product on  $X_{\psi}$  and  $Z_{\psi}$  (see *e.g.* Lemma 3.4), and the definition of  $\eta_0$  (cf. Definition 3.9) we find

$$I_0(\psi) = \int_{\gamma_0} \eta_0 = -\frac{(n+1)\psi}{|G|} \int_{|G|\cdot\widetilde{\gamma}_0} \theta_0 = -(n+1)\psi \int_{\widetilde{\gamma}_0} \theta_0.$$

For the computation of the latter integral, we use that the residue map and the tube map are mutual adjoint, and then perform an explicit computation:

$$I_0(\psi) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{-(n+1)\psi dz_1 \wedge \ldots \wedge dz_n}{F_{\psi}(1, z_1, \ldots, z_n)} = \sum_{k \geq 0} \frac{1}{((n+1)\psi)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}},$$

where the  $z_i = x_i/x_0$  are affine coordinates. This is the period integral used in Theorem 4.3, to prove that  $\eta_0$  trivializes  $f_*K_{\mathcal{Z}/\mathbb{D}_{\infty}}(\log)$ .

To the local system  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  there is an associated Picard–Fuchs equation. We make the change of variable  $z = \psi^{-(n+1)}$ , so that  $I_0$  becomes

$$I_0(z) = \sum_{k \ge 0} \frac{z^k}{(n+1)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}}.$$

Define the differential operators  $\delta = z \frac{d}{dz}$  and

(5.7) 
$$D = \delta^n - z \prod_{j=1}^n \left( \delta + \frac{j}{n+1} \right).$$

Differentiating  $I_0(z)$  term by term and repeteadly, one checks  $DI_0(z) = 0$ . Now, on the one hand D = 0 is a degree n irreducible differential equation of hypergeometric type [Kat90, Cor. 3.2.1]. On the other hand,  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  is a local system of rank n. It follows that D = 0 is necessarily the Picard–Fuchs equation satisfied by the periods of  $\eta_0$ .

We now exhibit all the solutions of the Picard–Fuchs equation. For dimension reasons, these will determine a multivalued basis of homology cycles. Following Zinger (see *e.g.* [Zin08, pp. 1214–1215]), for q = 0, ..., n-1 we define an *a priori* formal series  $I_{0,q}$  by

$$\sum_{q=0}^{\infty} I_{0,q}(t) w^q = e^{wt} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{r=1}^{(n+1)d} ((n+1)w + r)}{\prod_{r=1}^{d} (w+r)^{n+1}} =: R(w,t).$$

Let us also define F(w, t) for the infinite sum on the right hand side, so that  $R(w, t) = e^{wt}F(w, t)$ . Under the change of variable

(5.8) 
$$e^{t} = (n+1)^{-(n+1)}z = ((n+1)\psi)^{-(n+1)},$$

the series  $I_{0,0}(t)$  becomes  $I_0(z) = I_0(\psi)$  [Zin08, eq. (2–17)].

**Proposition 5.3.** Under the change of variable (5.8), the functions  $I_{0,q}(z)$ , q = 0, ..., n-1, define a basis of multivalued holomorphic solutions of the Picard-Fuchs equation for the local system  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  on 0 < |z| < 1.

*Proof.* After the change of variable, one checks that F(w,z) is absolutely convergent on compact subsets in the region |w| < 1 and |z| < 1. This implies that the functions  $I_{0,q}(z)$  are multivalued holomorphic functions on 0 < |z| < 1. Again taking into account the change of variable, it is

formal to verify that R(w, t) solves the Picard-Fuchs equation (5.7), and hence so do the functions  $I_{0,q}(z)$ . To see that they form a basis of solutions, it is enough notice that each  $I_{0,q}(z)$  has a singularity of the form  $(\log z)^q$  as  $z \to 0$ .

By the proposition, and because  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  has rank n, the functions  $I_{0,q}(z)$  determine a flat multivalued basis of sections  $\gamma_q$  of  $(R^{n-1}f_*\mathbb{C})_{\text{prim}}^{\vee}$  on 0 < |z| < 1, by the recipe

$$I_{0,q}(z) = \int_{\gamma_q(z)} \eta_0.$$

See for instance [Voi99, Sec. 3.4 & Lemme 3.12] for a justification. The notation is compatible with the invariant cycle  $\gamma_0$  constructed above, as we already observed that  $I_{0,0}(z) = I_0(z)$ . The flat multivalued basis elements  $\gamma_q(z)$  provide a basis of  $(H_{n-1})_{\text{lim}}$ , the limiting Hodge structure on the homology, at infinity. We still write  $\gamma_0, \ldots, \gamma_{n-1}$  for this limit basis. We next prove it is adapted to the weight filtration.

**Proposition 5.4.** Let  $W'_{\bullet}$  be the weight filtration of the limiting mixed Hodge structure on  $(H_{n-1})_{\lim}$ . Then  $\gamma_q \in W'_{2q} \setminus W'_{2q-1}$ .

*Proof.* It is enough to establish the analogous property for the Poincaré duals  $\gamma'_q \in H^{n-1}_{\lim}$ , similarly defined as the limits of the Poincaré duals  $\gamma'_q(z)$  of the  $\gamma_q(z)$ . On each fiber  $Z_z$ , the Hodge decomposition and the Cauchy–Schwarz inequality imply

$$|I_{0,q}(z)| = \left| \int_{Z_z} \gamma'_q(z) \wedge \eta_0 \right| \le (2\pi)^{n-1} \|\gamma'_q(z)\|_{L^2} \|\eta_0\|_{L^2}.$$

Now  $|I_{0,q}(z)|$  grows like  $(\log|z|^{-1})^q$  as  $z\to 0$  along angular sectors (cf. proof of Proposition 5.3). Because the monodromy is maximally unipotent at infinity and  $\eta_0$  is a basis of  $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ , the  $L^2$  norm  $\|\eta_0\|_{L^2}$  grows like  $(\log|z|^{-1})^{(n-1)/2}$  (see [EFiMM18b, Thm. A] or the more general [EFiMM18a, Thm. 4.4]). We infer that as  $z\to 0$ , along angular sectors,

$$\|\gamma_q'(z)\|_{L^2} \gtrsim (\log|z|^{-1})^{\frac{2q-(n-1)}{2}}.$$

By Schmid's metric characterization of the limiting Hodge structure [Sch73, Thm. 6.6], we then see that  $\gamma'_q \notin W_{2q-1}$ .

It remains to show that  $\gamma_q' \in W_{2q}$ . First of all, starting with q = n - 1, we already know  $\gamma_{n-1}' \in W_{2n-2} \setminus W_{2n-3}$ . We claim that  $\gamma_{n-2}' \in W_{2n-4}$ . Otherwise  $\gamma_{n-2}' \in W_{2n-2} \setminus W_{2n-4}$ . But the weight filtration has one-dimensional graded pieces in even degrees, and zero otherwise (cf. Lemma 4.1). It follows that  $W_{2n-4} = W_{2n-3}$  and  $\gamma_{n-1}' = \lambda \gamma_{n-2}' + \beta$ , for some constant  $\lambda$  and some  $\beta \in W_{2n-4}$ . Integrating against  $\eta_0$ , this relation entails

$$I_{0,n-1}(z) = \lambda I_{0,n-2}(z) + \int_{Z_z} \beta(z) \wedge \eta_0,$$

where  $\beta(z)$  is the flat multivalued section corresponding to  $\beta$ . Let us examine the asymptotic behaviour of the right hand side of this equality, as  $z \to 0$ , along angular sectors. We know that  $|I_{0,n-2}(z)|$  grows like  $(\log|z|^{-1})^{n-2}$ . By the Hodge decomposition and the Cauchy–Schwarz inequality, and Schimd's theorem, the integral grows at most like  $(\log|z|^{-1})^{n-2}$ . This contradicts that  $|I_{0,n-1}(z)|$  grows like  $(\log|z|^{-1})^{n-1}$ . Hence  $\gamma'_{n-2} \in W_{2n-4}$ . Continuing inductively in this fashion, we conclude that  $\gamma'_{q} \in W_{2q}$  for all q, as desired.

A normalized basis of  $\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$ . We construct a new basis of holomorphic sections of  $\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$  close to infinity, and corresponding period integrals  $I_{p,q}(z)$ . We proceed inductively:

- (1) set  $\tilde{\vartheta}_0 = \eta_0$ ;
- (2) for  $p \ge 1$ , suppose that  $\widetilde{\vartheta}_0, \dots, \widetilde{\vartheta}_{p-1}$  have been constructed. Define

$$I_{p-1,q}(z) = \int_{\gamma_q(z)} \widetilde{\vartheta}_{p-1}.$$

This notation is consistent with the previous definition of  $I_{0,q}$ ;

(3) assuming for the time being that  $I_{p-1,p-1}(z)$  is holomorphic and non-vanishing at z=0 (see the proof of Proposition 5.5 below), we define  $\widetilde{\vartheta}_p$  by

(5.9) 
$$\widetilde{\vartheta}_{p} = \nabla_{zd/dz} \left( \frac{\widetilde{\vartheta}_{p-1}}{I_{p-1,p-1}(z)} \right);$$

One verifies integrating (5.9) over  $\gamma_q(z)$  that the period integrals  $I_{p,q}(z) := \int_{\gamma_q(z)} \widetilde{\vartheta}_p$  satisfy the following recursion:

(5.10) 
$$I_{p,q}(z) = z \frac{d}{dz} \left( \frac{I_{p-1,q}(z)}{I_{p-1,p-1}(z)} \right).$$

Taking into account the change of variable (5.8), we see that this is the same recurrence relation as in [Zin08, eq. (2–18)] (see also [Zin09, eq. (0.16)]). Hence the  $I_{p,q}(z)$  above coincides with the  $I_{p,q}(t)$  in *loc. cit.* We further normalize:

$$\vartheta_p = \frac{\widetilde{\vartheta}_p}{I_{p,p}(z)}.$$

**Proposition 5.5.** (1) For all k, the sections  $\{\vartheta_j\}_{j=0,\dots,k}$ , constitute a holomorphic basis of the filtered piece  $\mathscr{F}^{n-1-k}R^{n-1}f_*\Omega^{\bullet}_{\mathscr{Z}/\mathbb{D}_{\infty}}(\log)_{\mathrm{prim}}$ .

(2) The periods of  $\vartheta_k$  satisfy

$$\int_{\gamma_k} \vartheta_k = 1 \quad and \quad \int_{\gamma_q} \vartheta_k = 0 \quad if \quad q < k.$$

(3) The projection of  $\vartheta_k$  to  $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_{\infty}}^{n-1-k}(\log)_{\text{prim}}$  relates to  $\eta_k$  by

$$(\vartheta_k)^{n-1-k,k} = \frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

(4) The sections  $\{\vartheta_i\}_{i=0,\dots,n-1}$  are uniquely determined by properties (1)–(2) above.

*Proof.* We notice that the period integrals  $I_{p,p}(z)$  are holomorphic in z and non-vanishing at z=0. This is [Zin09, Prop. 3.1], in turn based on [ZZ08]. With this observation at hand, the claims (1)–(3) then follow from properties of the Gauss–Manin connection and Kodaira–Spencer maps, Lemma 3.11 and Theorem 4.3. The details are left to the reader. The uniqueness property is obtained by comparing two such bases adapted to the Hodge filtration as in (1), and then imposing the period relations (2).

Actually, the basis  $\vartheta_{\bullet} = \{\vartheta_j\}_{j=0,\dots,n-1}$  is determined by the limiting Hodge structure  $H_{\lim}^{n-1}$ , up to constant, as we now show:

- (1) Let  $\gamma'_{\bullet}$  be an adapted basis of the weight filtration on  $(H_{n-1})_{\lim}$ , as in **Proposition 5.6.** Proposition 5.4. Then there exists a unique holomorphic basis  $\vartheta'_{\bullet}$  of  $R^{n-1}f_*\Omega^{\bullet}_{\mathcal{T}/\mathbb{D}}$  (log)<sub>prim</sub> satisfying the conditions analogous to (1)–(2) with respect to  $\gamma'_{\bullet}$ .
  - (2) There exist non-zero constants  $c_k \in \mathbb{C}$  such that  $\vartheta'_k = c_k \vartheta_k$ .

*Proof.* We prove both assertions simultaneously. We write  $\gamma_{\bullet}$  and  $\gamma'_{\bullet}$  as column vectors. Since the graded pieces of the weight filtration on  $(H_{n-1})_{lim}$  are all one-dimensional, there exists a lower triangular matrix  $A \in GL_n(\mathbb{C})$  with  $\gamma'_{\bullet} = A\gamma_{\bullet}$ . If we decompose A = D + L, where D is diagonal and L is lower triangular, we see that the entries of the column vector  $\vartheta'_{\bullet} := D^{-1}\vartheta_{\bullet}$ fulfill the requirements.

**Definition 5.7.** We define the canonical trivializing section of  $R^k f_* \Omega_{\mathcal{I}/\mathbb{D}_{\infty}}^{n-1-k}(\log)_{\text{prim}}$  to be

$$\widetilde{\eta}_k = (\vartheta_k)^{n-1-k,k} = \frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

By the previous proposition, up to constants, the sections  $\tilde{\eta}_k$  depend only on  $(H_{n-1})_{\text{lim}}$ , or equivalently  $H_{\text{lim}}^{n-1}$  by Poincaré duality. For a general discussion about distinguished sections, we refer the reader to [Mor97, Section 6.3].

5.3. Generating series of Gromov-Witten invariants and Zinger's theorem. In order to state Zinger's theorem on generating series of Gromov-Witten invariants of genus one, and for coherence with the notations of this author, it is now convenient to work in the t variable instead of z. The mirror map in Zinger's normalizations is the change of variable

$$t \mapsto T = \frac{I_{0,1}(t)}{I_{0,0}(t)} = \frac{\int_{\gamma_1(t)} \eta_0}{\int_{\gamma_0(t)} \eta_0}.$$

Notice that this differs by a factor  $2\pi i$  from the more standard Morrison's mirror map [Mor93] used in the introduction. The Jacobian of the mirror map is computed from (5.10)

$$\frac{dT}{dt} = I_{1,1}(t).$$

Let us introduce some last notations:

- $X_{n+1}$  denotes a general degree n+1 hypersurface in  $\mathbb{P}^n$ .  $N_1(0) = -\left(\frac{(n-1)(n+2)}{48} + \frac{1-(-n)^{n+1}}{24(n+1)^2}\right) = \frac{1}{24}\left(-\frac{n(n+1)}{2} + \frac{\chi(X_{n+1})}{n+1}\right)$ .
- $N_1(d)$  is the genus 1 and degree d Gromov-Witten invariant of  $X_{n+1}$   $(d \ge 1)$ .

From these invariants we build a generating series:

(5.11) 
$$F_1^A(T) = N_1(0)T + \sum_{d=1}^{\infty} N_1(d)e^{dT}.$$

It follows from [Zin08, Thm. 2] that this generating series satisfies

$$\begin{split} F_1^A(T) &= N_1(0)\,t + \frac{(n+1)^2 - 1 + (-n)^{n+1}}{24(n+1)} \log I_{0,0}(t) \\ &- \begin{cases} \frac{n}{48} \log(1 - (n+1)^{n+1}e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8} \log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48} \log(1 - (n+1)^{n+1}e^t) + \sum_{p=0}^{(n-3)/2} \frac{(n+1-2p)(n-1-2p)}{8} \log I_{p,p}(t) & \text{if } n \text{ odd} \,. \end{cases} \end{split}$$

This identity has to be understood in the sense of formal series. As an application of relations between the hypergeometric series  $I_{p,p}(t)$ , studied in detail in [ZZ08], the following identity holds (for a version of this particular identity, see [Zin09, eq. (3.2)]):

$$\frac{n(3n-5)}{48}\log(1-(n+1)^{n+1}e^t) + \frac{1}{2}\sum_{p=0}^{n-2} \binom{n-p}{2}\log I_{p,p}(t) = \\ \begin{cases} \frac{n}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8}\log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-3)/2} \frac{(n+1-2p)(n-1-2p)}{8}\log I_{p,p}(t) & \text{if } n \text{ odd} \end{cases}$$

Consequently, Zinger's theorem takes the following pleasant form, that we will use to simplify the task of recognizing  $F_1^A(T)$  in our expression for the BCOV invariant (cf. Theorem 5.1).

**Theorem 5.8** (Zinger). *Under the change of variables*  $t \mapsto T$ , the series  $F_1^A(T)$  takes the form

(5.12) 
$$F_1^A(T) = N_1(0)t + \frac{\chi(X_{n+1})}{24} \log I_{0,0}(t) - \frac{n(3n-5)}{48} \log(1 - (n+1)^{n+1}e^t) - \frac{1}{2} \sum_{p=0}^{n-2} {n-p \choose 2} \log I_{p,p}(t).$$

A final remark on the holomorphicity of  $F_1^A(T)$  is in order. While Theorem 5.8 is *a priori* an identity of formal series, the right hand side of (5.12) is actually a holomorphic function in t, for Re  $t \ll 0$ . Then, via the mirror map,  $F_1^A(T)$  acquires the structure of a holomorphic function in T. One can check that the domain of definition is a half-plane Re  $T \ll 0$ .

5.4. **Genus one mirror symmetry and the BCOV invariant.** We are now in position to show that the BCOV invariant of the mirror family  $f \colon \mathcal{Z} \to U$  realizes genus one mirror symmetry for Calabi–Yau hypersurfaces in projective space. That is, one can extract the generating series  $F_1^A(T)$  from the function  $\psi \mapsto \tau_{BCOV}(Z_\psi)$ . The precise recipe by which this is accomplished goes through expressing  $\tau_{BCOV}$  in terms of the  $L^2$  norms of the canonical sections  $\widetilde{\eta}_k$  (cf. Definition 5.7). But first we need to make  $\tau_{BCOV}(Z_\psi)$  and  $F_1^A(T)$  depend on the same variable. To this end, we let

(5.13) 
$$F_1^B(\psi) = F_1^A(T), \quad \text{for} \quad T = \frac{I_{0,1}(t)}{I_{0,0}(t)} \quad \text{and} \quad e^t = ((n+1)\psi)^{-(n+1)}.$$

**Theorem 5.9.** In a neighborhood of  $\psi = \infty$ , there is an equality

$$\tau_{BCOV}(Z_{\psi}) = C \left| \exp\left( (-1)^{n-1} F_1^B(\psi) \right) \right|^4 \frac{\|\widetilde{\eta}_0\|_{L^2}^{\chi/6}}{\left( \prod_{k=0}^{n-1} \|\widetilde{\eta}_k\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}},$$

where 
$$\chi = \chi(Z_{\psi})$$
 and  $C \in \pi^{c} \mathbb{Q}_{>0}^{\times}$ ,  $c = \frac{1}{2} \sum_{k} (-1)^{k+1} k^{2} b_{k}$ .

*Proof.* The proof is a simple computation, which consists in changing the variable T to  $\psi$ , using (5.13), in the expression for  $F_1^A(T)$  provided by Theorem 5.8. Modulo log of rational numbers,

we find

$$\begin{split} 4F_1^A(T) &= \left(-\frac{n(n+1)}{12} + \frac{\chi(X_{n+1})}{6(n+1)}\right)t + \frac{\chi(X_{n+1})}{6}\log I_{0,0}(t) \\ &- \frac{n(3n-5)}{12}\log(1-(n+1)^{n+1}e^t) - 2\sum_{p=0}^{n-2}\binom{n-p}{2}\log I_{p,p}(t) \\ &= \left(\frac{n(n+1)}{12} - \frac{\chi(X_{n+1})}{6(n+1)} + \frac{n(3n-5)}{12}\right)\log(\psi^{n+1}) \\ &- \frac{n(3n-5)}{12}\log(\psi^{n+1}-1) + \frac{\chi(X_{n+1})}{6}\log I_{0,0}(t) - 2\sum_{p=0}^{n-2}\binom{n-p}{2}\log I_{p,p}(t) \\ &= (-1)^{n-1}\log\frac{(\psi^{n+1})^{2a}}{(\psi^{n+1}-1)^{2b}} + \frac{\chi(X_{n+1})}{6}\log I_{0,0}(t) - 2\sum_{p=0}^{n-2}\binom{n-p}{2}\log I_{p,p}(t). \end{split}$$

Now, in terms of the canonical trivializing sections  $\tilde{\eta}_k$  given in Definition 5.7, Theorem 5.1 becomes:

$$\tau_{BCOV}(Z_{\psi}) = C \left| \frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b} \right|^2 \frac{|I_{0,0}(t)|^{\chi/6}}{\left(\prod_{p=0}^{n-2} |I_{p,p}(t)|^{2\binom{n-p}{2}}\right)^{(-1)^{n-1}}} \frac{\|\widetilde{\eta}_0\|_{L^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\widetilde{\eta}_k\|_{L^2}^{2(n-1-k)}\right)^{(-1)^{n-1}}}.$$

- *Remark* 5.10. (1) In relative dimension 3, we recover the main theorem of Fang–Lu–Yoshikawa [FLY08, Thm 1.3]. Their result is presented in a slightly different form. The first formal discrepancy is in the choice of the trivializing sections. Their trivializations can be related to ours via Kodaira–Spencer maps. The second discrepancy is explained by a different normalization of  $F_1^A$ : they work with two times Zinger's generating series. This justifies why their expression for the BCOV invariant contains  $|\exp(-F_1^B(\psi))|^2$ , while our formula in dimension 3 specializes to  $|\exp(-F_1^B(\psi))|^4$ .
  - (2) The norms of the sections  $\tilde{\eta}_k$  are independent of the choice of crepant resolution. It follows that the expression on the right hand side in Theorem 5.9 is independent of the crepant resolution, except possibly for the constant C. In [EFiMM18a, Conj. B] we conjectured that the BCOV invariant is a birational invariant. Thus C should in fact be independent of the choice of crepant resolution.

#### **Corollary 5.11.** (1) The invariant $N_1(0)$ satisfies

$$N_1(0) = \frac{-1}{24} \int_{X_{n+1}} c_{n-2}(X_{n+1}) \wedge H,$$

where H is the hyperplane class in  $\mathbb{P}^n$ .

(2) As  $\psi \to \infty$ ,  $\log \tau_{BCOV}(Z_{\psi})$  behaves as

(5.14) 
$$\log \tau_{BCOV}(Z_{\psi}) = \left(\frac{(-1)^n}{12} \int_{X_{n+1}} c_{n-2}(X_{n+1}) \wedge H\right) \log \left|\psi^{-(n+1)}\right|^2 + O(\log\log|\psi|).$$

*Proof.* The sought for interpretation of  $N_1(0)$ , or equivalently for the coefficient  $\kappa_{\infty}$  in Corollary 5.2, is obtained by an explicit computation of, and comparison to  $\int_{X_{n+1}} c_{n-1}(\Omega_{X_{n+1}}) \wedge H$ . Indeed,

by the cotangent exact sequence for the immersion of  $X_{n+1}$  into  $\mathbb{P}^n$ , this reduces to

$$\int_{X_{n+1}} \mathsf{c}_{n-2}(\Omega_{X_{n+1}}) \wedge H = \frac{(-1)^{n-1}}{n+1} \chi(X_{n+1}) - \int_{\mathbb{P}^n} \mathsf{c}_{n-1}(\Omega_{\mathbb{P}^n}) \wedge H,$$

and we have explicit formulas for both terms on the right. This settles both the first and second claims.  $\Box$ 

*Remark* 5.12. The asymptotic expansion (5.14) has been written in the variable  $\psi^{-(n+1)}$  on purpose, since this is the natural parameter in a neighborhood of the MUM point in the moduli space. In this form, the equation agrees with the predictions of genus one mirror symmetry (cf. [EFiMM18a, Sec. 1.4] for a discussion).

#### 6. The refined BCOV conjecture

In this section, we propose an alternative approach to genus one mirror symmetry for Calabi–Yau manifolds, which bypasses spectral theory and is closer in spirit to the genus zero picture. The counterpart of the Yukawa coupling on the mirror side will now be a Grothendieck–Riemann–Roch isomorphism (GRR) of line bundles, built out of Hodge bundles. As in the case of the Yukawa coupling, these Hodge bundles should be trivialized in a canonical way for maximally unipotent degenerations, and the expression of the GRR isomorphism in these trivializations should then encapsulate the genus one Gromov–Witten invariants of the original Calabi–Yau manifold. This is our interpretation of the holomorphic limit of the BCOV invariant. We refer to this conjectural program as *the refined BCOV conjecture*, which is divided into two parts and described below.

*The Grothendieck–Riemann–Roch isomorphism*. Let  $f: X \to S$  be a projective morphism of connected complex manifolds, whose fibers are Calabi–Yau manifolds. Recall from (2.6) that the BCOV bundle  $\lambda_{BCOV}(X/S)$  is defined as a combination of determinants of Hodge bundles. Its formation commutes with arbitrary base change.

**Conjecture 1.** For every projective family of Calabi–Yau manifolds  $f: X \to S$  as above, there exists a natural isomorphism of line bundles, compatible with any base change,

**GRR**(X/S): 
$$\lambda_{BCOV}(X/S)^{\otimes 12\kappa} \xrightarrow{\sim} f_*(K_{X/S})^{\otimes \chi\kappa}$$
.

Here  $\chi$  is the Euler characteristic of any fiber of f and  $\kappa$  only depends on the relative dimension of f.

Let us present arguments in favour of the conjecture:

- applying this to the universal elliptic curve, the right hand side becomes trivial in view of  $\chi=0$ . This suggests that the left hand side is trivial. It is indeed trivialized by the discriminant modular form  $\Delta$ , with  $\kappa=1$ . For higher dimensional abelian varieties both sides are trivial and the identity provides a natural isomorphism.
- for *K*3 surfaces both sides are identitical, and the identity provides a natural isomorphism. For Enriques surfaces a result similar to that of elliptic curves exist, see [Pap08]. This can probably also be realized by a Borcherds product [Bor96].
- in the category of schemes, a natural isomorphism of  $\mathbb{Q}$ -line bundles up to sign exists by work of Franke [Fra92] and the first author [Eri08]. It is compatible with the arithmetic Riemann–Roch theorem, but is far more general and stronger.

The following proposition establishes a version of Conjecture 1 in the setting of arithmetic varieties (cf. Section 2.3). This is an application of the arithmetic Riemann–Roch theorem 2.3. Recall that an arithmetic ring A comes together with a finite collection of complex embeddings  $\Sigma$ , closed under complex conjugation. We will write  $A^{\times,1}$  for the group of elements  $u \in A^{\times}$  with  $|\sigma(u)| = 1$  for all embedding  $\sigma \in \Sigma$ . For instance, if A is the ring of integers of a number field then  $A^{\times,1}$  is a finite group. If  $A = \mathbb{Q}$  or  $\mathbb{R}$ , then  $A^{\times,1} = \{\pm 1\}$ . If  $A = \mathbb{C}$ , then  $A^{\times,1}$  is the unit circle in  $\mathbb{C}$ .

**Proposition 6.1.** Let  $f: X \to S$  be a smooth projective morphism of arithmetic varieties over an arithmetic ring A, with Calabi–Yau fibers. Let  $X_{\infty}$  be the generic fiber of f and write  $\chi = \chi(X_{\infty})$ . Assume that  $S \to \operatorname{Spec} A$  is surjective and has geometrically connected fibers. Then:

(1) there exists an integer  $\kappa \ge 1$  and an isomorphism of line bundles on S

**GRR**: 
$$\lambda_{BCOV}(X/S)^{\otimes 12\kappa} \xrightarrow{\sim} (f_*K_{X/S})^{\otimes \chi\kappa}$$
,

with the property of being an isometry for the Quillen-BCOV and  $L^2$  metrics on  $\lambda_{BCOV}(X/S)$  and  $f_*K_{X/S}$ , respectively.

(2) if **GRR**' is another such isomorphism, for another choice of integer  $\kappa' \ge 1$ , then

$$\mathbf{GRR}' \otimes \kappa = \mathbf{GRR} \otimes \kappa'$$

up to multiplication by some  $u \in A^{\times,1}$ . Consequently, the formation of **GRR** is compatible with any base change between geometrically connected arithmetic varieties over A, up to the power  $\kappa$  and multiplication by a unit in  $A^{\times,1}$ .

*Proof.* The first claim is a restatement of the identity (2.9) in  $\widehat{\operatorname{CH}}^1(S)_{\mathbb{Q}}$ , together with the isomorphism  $\widehat{\operatorname{c}}_1:\widehat{\operatorname{Pic}}(S)\stackrel{\sim}{\to}\widehat{\operatorname{CH}}^1(S)$  and the very definition of  $\widehat{\operatorname{Pic}}(S)$  as the group of isomorphism classes of hermitian line bundles over S.

For the second claim, notice that both  $\mathbf{GRR}'^{\otimes \kappa}$  and  $\mathbf{GRR}^{\otimes \kappa'}$  induce isometries between the hermitian line bundles  $\lambda_{BCOV}(X/S)^{\otimes 12\kappa\kappa'}$  and  $(f_*K_{X/S})^{\otimes \chi\kappa\kappa'}$ , endowed with the Quillen-BCOV and  $L^2$  metrics, respectively. These isomorphisms differ by multiplication by a unit  $u \in \Gamma(S, \mathcal{O}_S^{\kappa})$ . The isometry property guarantees that the induced holomorphic function on  $S^{\mathrm{an}}$  has modulus one. Since  $S \to \mathrm{Spec}\,A$  is surjective and has geometrically connected fibers, we necessarily have  $u \in A^{\kappa}$ . Now u has modulus one as a function on  $S^{\mathrm{an}}$ , which exactly means  $u \in A^{\kappa,1}$ . The base change property then follows from the compatibility of  $\lambda_{BCOV}(X/S)$  and  $f_*K_{X/S}$  with base change, and the fact that the Quillen and Hodge metrics are preserved as well.

*Remark* 6.2. (1) If  $A^{\times,1}$  is a finite group of order d, then the second claim of the corollary entails

$$\mathbf{GRR}' \otimes d\kappa = \mathbf{GRR} \otimes d\kappa'$$
.

Therefore, after possibly adjusting  $\kappa$ , the isomorphism is uniquely determined.

(2) The proposition applies to the mirror family of Calabi–Yau hypersurfaces studied in Section 3. Here  $A = \mathbb{Q}$ , and therefore the resulting isomorphism is determined by the previous remark.

Relationship with mirror symmetry. Our second conjecture suggests that for degenerating families of Calabi–Yau manifolds, with maximally unipotent monodromy, **GRR** realizes genus one mirror symmetry. In this section we formulate in general and prove it in the case of mirrors of hypersurfaces in projective space as a consequence of our previous main theorems. The case of K3 surfaces is not covered by those considerations, but a proof is also provided in this case.

**Conjecture 2.** Let  $f: X \to \mathbf{D}^{\times} = (\mathbb{D}^{\times})^d$  be a projective morphism of Calabi–Yau n-folds, with  $d = h^{1,n-1}$  the dimension of the deformation space of the fibers, effectively parametrized with maximally unipotent monodromy. Then there exist

- (1) canonical holomorphic coordinates  $\mathbf{q} = (q_1, ..., q_d)$  on  $\mathbf{D}$  (exponential mirror map),
- (2) canonical trivializations of the line bundles  $\lambda_{BCOV}(X/\mathbf{D}^{\times})$  and  $f_*K_{X/\mathbf{D}^{\times}}$ ,

such that  $GRR(X/\mathbf{D}^{\times})$  computed in this trivializations and in coordinate **q** becomes

$$\mathbf{GRR}(\mathbf{q}) = \exp\left((-1)^n F_1^A(\mathbf{q})\right)^{24\kappa},$$

where

$$F_1^A(\mathbf{q}) = -\frac{1}{24} \sum_{k=1}^d \left( \int_{X^{\vee}} \mathbf{c}_{n-1}(X^{\vee}) \wedge \omega_k \right) \log q_k + \sum_{\beta \in H_2(X^{\vee}, \mathbb{Z})} \mathrm{GW}_1(X^{\vee}, \beta) \; \mathbf{q}^{\langle \underline{\omega}, \beta \rangle}$$

is a generating series of genus one Gromov–Witten invariants on a mirror Calabi–Yau manifold  $X^{\vee}$ :

- $\omega = (\omega_1, ..., \omega_d)$  is some basis of  $H^{1,1}(X^{\vee}) \cap H^2(X^{\vee}, \mathbb{Z})$  formed by ample classes.
- $\overline{\mathrm{GW}}_1(X^{\vee},\beta)$  is the genus one Gromov–Witten invariant on  $X^{\vee}$  associated to the class  $\beta$ .
- $\mathbf{q}^{\langle \underline{\omega}, \beta \rangle} = \prod_k q_k^{\langle \omega_k, \beta \rangle}$ .

As supporting evidence, we consider the case of the mirror family of Calabi–Yau hypersurfaces in  $\mathbb{P}^n$ :

**Theorem 6.3.** Let  $n \ge 4$ . Then Conjecture 1 and Conjecture 2 are true, up to a constant, for the mirror family  $f: \mathcal{Z} \to \mathbb{D}_{\infty}^{\times}$  in a neighborhood of the MUM point.

*Proof.* First of all, the existence of a natural isomorphism as in Conjecture 1 is provided by Proposition 6.1 and Remark 6.2 (2). Secondly, for Conjecture 2, we need to trivialize the BCOV bundle  $\lambda_{BCOV}(\mathcal{Z}/\mathbb{D}_{\infty}^{\times})$ . We fix a polarization induced from a projective factorization of  $f:\mathcal{Z}\to\mathbb{P}^1$ . We denote by L the corresponding Lefschetz operator on  $R^{\bullet}f_*\mathbb{Q}$ . We have previously, in Definition 5.7, constructed sections  $\widetilde{\eta}_k$  of the primitive part of  $R^kf_*\Omega_{\mathcal{Z}/\mathbb{D}_{\infty}}^{n-1-k}$ . It was characterized by having lower diagonal unipotent period matrix with respect to a fixed flag on the limiting mixed Hodge structure. We proceed to set:

- If  $p+q\neq n-1$ , p=q then  $\widetilde{\eta}_{p,p}$  is a generator of the trivial  $\mathbb{Z}$ -local system  $(R^{2p}f_*\mathbb{Z})_{nt}$ .
- If p + q = n 1, and  $p \neq q$  then  $\widetilde{\eta}_{p,q} = \widetilde{\eta}_q$ .
- If p + q = n 1, and p = q we set  $\widetilde{\eta}_{p,p} = \widetilde{\eta}_p \otimes L\widetilde{\eta}_{p-1,p-1}$ .

We then define

$$\widetilde{\eta}_{BCOV} = \bigotimes_{p,q} \widetilde{\eta}_{p,q}^{\otimes (-1)^{p+q} p}.$$

Computing the Grothendieck–Riemann–Roch isomorphism **GRR** in the trivalizations  $\widetilde{\eta}_{BCOV}^{12\kappa}$  and  $\widetilde{\eta}_{n-1,0}^{\chi\kappa}$ , means considering the invertible holomorphic function

$$GRR(\mathbf{q}) = \frac{GRR(\widetilde{\eta}_{BCOV}^{12\kappa})}{\widetilde{\eta}_{n-1,0}^{\chi\kappa}},$$

where  $\mathbf{q}$  is the mirror coordinate on  $\mathbf{D}_{\infty}$ . By the isometry property of **GRR** and the very definition of the BCOV invariant, we have

$$\tau_{BCOV}^{12\kappa} = \frac{\|\mathbf{GRR}(\widetilde{\eta}_{BCOV}^{12\kappa})\|_{L^{2}}^{2}}{\|\widetilde{\eta}_{BCOV}\|_{L^{2}BCOV}^{24\kappa}}.$$

In other words,

(6.1) 
$$\tau_{BCOV} = |\mathbf{GRR}(\mathbf{q})|^{1/6\kappa} \frac{\|\widetilde{\eta}_{n-1,0}\|_{L^2}^{\chi/6}}{\|\widetilde{\eta}_{BCOV}\|_{L^2,BCOV}^2}.$$

As in the proof of Theorem 5.1 (see also [EFiMM18a, Prop. 4.2]), the quantity  $\|\widetilde{\eta}_{BCOV}\|_{L^2,BCOV}$  coincides with the factor  $\prod_{k=0}^{n-1} \|\widetilde{\eta}_k\|_{L^2}^{n-1-k}$  up to a constant. We conclude by comparing (6.1) with Theorem 5.9.

The cases of one and two dimensional Calabi–Yau varieties are not covered by the above result. The one dimensional case essentially corresponds to the Kronecker limit formula recalled in §1.5. We now study the case of K3 surfaces. Since  $h^{1,1}=20$  for a K3 surface, our one-dimensional Dwork-type family cannot be a mirror family. It is still expected that the mirror of a K3 surface is a K3 surface, a systematic construction in terms of polarized lattices can be found in for example [Dol96]. We will assume this below.

**Proposition 6.4.** Conjecture 1 and Conjecture 2 are true, up to a constant, for any mirror family of a K3 surface. Moreover,  $\kappa = 1$ .

*Proof.* The BCOV bundle takes a particularly simple form for a *K*3 surface *X*, its square can be written as:

$$(6.2) \lambda_{BCOV}(X)^{\otimes 2} = \det H^{2,0}(X)^{\otimes 4} \otimes \det H^{1,1}(X)^{\otimes 4} \otimes \det H^{2,2}(X)^{\otimes 4} \simeq \det H^{2,0}(X)^{\otimes 4}$$

The isomorphisms  $\det H^{1,1}(X)^{\otimes 2} \simeq \mathbb{C}$  and  $\det H^{2,2}(X) \simeq \mathbb{C}$  are both induced by Serre duality and are thus isometries, for the  $L^2$  norms and standard metric on  $\mathbb{C}$ . Since  $\chi(X) = 24$ , the square of the right hand side of Conjecture 1 is provided by the same object.

Let  $f: \mathcal{X} \to \mathbf{D}^{\times}$  be a family of K3 surfaces. The previous construction globalizes to an isomorphism of line bundles

$$\lambda_{BCOV}(\mathcal{X}/\mathbf{D}^{\times})^{\otimes 2} \xrightarrow{\sim} (f_* K_{\mathcal{X}/\mathbf{D}^{\times}})^{\otimes 4}$$

compatible with base change. Taking 6th powers and setting  $\kappa = 1$ , this proves Conjecture 1 in this case. We hence propose that **GRR** is induced by (6.2).

Following the proof of Theorem 6.3, prove Conjecture 2, we need to fabricate natural sections of both sides. We choose a section of  $\det R^1 f_* \Omega^1_{\mathcal{X}/\mathbf{D}^\times} = \det R^2 f_* \mathbb{C} \otimes \mathcal{O}_{\mathbf{D}^\times}$  induced by a generator of  $\det R^2 f_* \mathbb{Z}$  as a local system of  $\mathbb{Z}$ -modules, and analogously for  $\det R^2 f_* \Omega^2_{\mathcal{X}/\mathbf{D}^\times} = \det R^4 f_* \mathbb{C} \otimes \mathcal{O}_{\mathbf{D}^\times}$ . Their  $L^2$  norms are locally constant by [EFiMM18a, Prop. 4.2], for the implicit polarization coming from the projective assumption of Conjecture 2. Picking any section  $\widetilde{\eta}_{2,0}$  of  $f_* K_{\mathcal{X}/\mathbf{D}^\times}$ , it allows us to write down a natural section  $\widetilde{\eta}_{BCOV}$  of  $\lambda_{BCOV}(\mathcal{X}/\mathbf{D}^\times)$ .

The analogous formula to (6.1) becomes, in this case,

$$\tau_{BCOV} = |\mathbf{GRR}(\mathbf{q})|^{1/6} \frac{\|\widetilde{\eta}_{2,0}\|_{L^2}^4}{\|\widetilde{\eta}_{BCOV}\|_{L^2,BCOV}^2} = C|\mathbf{GRR}(\mathbf{q})|^{1/6},$$

for a constant C > 0. By triviality of the Gromov–Witten invariants for K3 surfaces (see for example [LP07, Corollary 3.3]), to prove Conjecture 2 we need to prove that  $\tau_{BCOV}$  is constant. This is the content of [EFiMM18a, Thm. 5.12].

#### 7. A CHOWLA-SELBERG FORMULA FOR THE BCOV INVARIANT

In this section we discuss an example of use of the arithmetic Riemann–Roch theorem to evaluate the BCOV invariant of a Calabi–Yau manifold with complex multiplication, similar to the derivation of the Chowla–Selberg formula from the Kronecker limit formula for elliptic curves. In such situations, or more generally for Calabi–Yau manifolds whose Hodge structures have some extra symmetries, we expect that the BCOV invariant can be evaluated in terms of special values of  $\Gamma$  functions or other special functions.

Let  $p \ge 5$  be a prime number, and define n = p - 1. We consider the mirror family  $f: \mathcal{Z} \to U$  to Calabi–Yau hypersurfaces of degree p in  $\mathbb{P}^n$ . The restriction on the dimension here has been made to simplify the exposition. The special fiber  $Z_0$  is a crepant resolution of  $X_0/G$ , where  $X_0$  is now the Fermat hypersurface

$$x_0^p + \ldots + x_n^p = 0.$$

The quotient  $X_0/G$  has an extra action of  $\mu_p \subset \mathbb{C}$ : a p-th root of unity  $\xi \in \mathbb{C}$  sends a point  $(x_0: \ldots: x_n)$  to  $(x_0: \ldots: x_{n-1}: \xi x_n)$ . This action induces a  $\mathbb{Q}$ -linear action of  $K = \mathbb{Q}(\mu_p) \subset \mathbb{C}$  on  $H^{n-1}(X_0,\mathbb{Q})^G$ . As a rational Hodge structure, the latter is isomorphic to  $H^{n-1}(Z_0,\mathbb{Q})$  (cf. §3.2 and Proposition 3.7; all the cohomology is primitive now). Hence  $H^{n-1}(Z_0,\mathbb{Q})$  inherits a  $\mathbb{Q}$ -linear action of K. Observe that  $[K:\mathbb{Q}] = p-1$ , which is exactly the dimension of  $H^{n-1}(Z_0,\mathbb{Q})$ . We say that  $Z_0$  has complex multiplication by K. Similarly, the algebraic de Rham cohomology  $H^{n-1}(Z_0,\Omega^{\bullet}_{Z_0/\mathbb{Q}})$  affords a  $\mathbb{Q}$ -linear action of K. Indeed, this is clear for  $H^{n-1}(X_0,\Omega^{\bullet}_{X_0/\mathbb{Q}})^G$ , since the action of  $\mu_p$  on  $X_0$  by automorphisms can actually be defined over  $\mathbb{Q}$  and commutes with the G action. Then, we transfer this to  $Z_0$  via Proposition 3.7 and Lemma 3.12.

Let us fix a non-trivial  $\xi \in \mu_p$ . If we base change  $H^{n-1}(Z_0,\mathbb{Q})$  to K, we have an eigenspace decomposition

$$H^{n-1}(Z_0,K) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0,K)_{\xi^k}.$$

Hence,  $\xi$  acts by multiplication by  $\xi^k$  on  $H^{n-1}(Z_0,K)_{\xi^k}$ . Similarly, for algebraic de Rham cohomology:

$$H^{n-1}(Z_0, \Omega_{Z_0/K}^{\bullet}) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0, \Omega_{Z_0/K}^{\bullet})_{\xi^k}.$$

If we compare with  $H^{n-1}(X_0,\Omega_{X_0/K}^{\bullet})^G$ , and we recall the construction of the sections  $\theta_k$  and  $\eta_k^{\circ}$  (cf. §3.3), we see by inspection that  $\xi$  acts on  $\eta_k^{\circ}$  by multiplication by  $\xi^{k+1}$ . Therefore, we infer that the non-trivial eigenspaces only occur when  $1 \le k \le p-1$  and

$$H^{n-1}(Z_0, \Omega_{Z_0/K}^{\bullet})_{\xi^k} = K\eta_{k-1}^{\circ} = H^{k-1}(Z_0, \Omega_{Z_0/K}^{n-k}).$$

Hence, the eigenspace  $H^{n-1}(Z_0, \Omega^{\bullet}_{Z_0/K})_{\xi^k}$  has Hodge type (n-k, k-1).

The period isomorphism relating algebraic de Rham and Betti cohomologies decomposes into eigenspaces as well. We obtain refined period isomorphisms

$$\operatorname{per}_k \colon H^{n-1}(Z_0, \Omega^{\bullet}_{Z_0/K})_{\xi^k} \otimes_K \mathbb{C} \xrightarrow{\sim} H^{n-1}(Z_0, K)_{\xi^k} \otimes_K \mathbb{C}.$$

Evaluating the isomorphism on K-bases of both sides, we obtain a period, still denoted  $\operatorname{per}_k \in \mathbb{C}^\times/K^\times$ .

**Lemma 7.1.** Fix an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$  in  $\mathbb{C}$ . Then there is an equality in  $\mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ 

$$\operatorname{per}_k = \frac{1}{\pi} \Gamma \left( \frac{k+1}{p} \right)^p.$$

*Proof.* The claim is equivalent to the analogous computation on  $X_0$ . Hidden behind this phrase is the comparison of cup products on  $X_0$  and  $Z_0$  accounted for by Lemma 3.4. On  $X_0$ , the formula for the period is well-known, and given for instance in Gross [Gro78, Sec. 4, p. 206] (see more generally [DMOS82, Chap. I, Sec. 7]). Notice that the author would rather work with the Fermat hypesurface  $x_0^p + \ldots + x_{n-1}^p = x_n^p$ . However, as we compute periods up to algebraic numbers, by applying the obvious isomorphism of varieties defined over  $\overline{\mathbb{Q}}$ , the result is the same. Also, we have used standard properties of the Γ-function to transform *loc. cit.* in our stated form.

**Theorem 7.2.** For  $Z_0$  of dimension p-2, with  $p \ge 5$  prime, the BCOV invariant satisfies

$$\tau_{BCOV}(Z_0) = \frac{1}{\pi^{\sigma}} \left( \Gamma \left( \frac{1}{p} \right)^{\chi/12} \prod_{k=1}^{p-1} \Gamma \left( \frac{k}{p} \right)^{p-k-1} \right)^{2p} \quad in \quad \mathbb{R}^{\times} / \mathbb{R} \cap \overline{\mathbb{Q}}^{\times},$$

where

$$\sigma = 3(p-2)\left(\frac{\chi}{12} + \frac{(p-1)(p-2)}{2}\right) + \frac{1}{4}\sum_{k}(-1)^{k}k^{2}b_{k}.$$

*Proof.* We apply Theorem 5.1, written in terms of the sections  $\eta_k^{\circ}$  instead of  $\eta_k$  (which vanish at 0). Up to rational number, this has the effect of letting down the term  $(\psi^{n+1})^a$  in that statement. We are thus lead to evaluated the  $L^2$  norms of the sections  $\eta_k^{\circ}$ . By [MR04, Lemma 3.4], the  $L^2$  norms satisfy

$$\|\eta_k^{\circ}\|_{L^2}^2 = (2\pi)^{-(p-2)} |\operatorname{per}_k|^2.$$

It is now enough to plug this expression in Theorem 5.1, as well as the value of  $\operatorname{per}_k$  provided by Lemma 7.1.

Combining Theorem 2.3 and the conjecture of Gross–Deligne (cf. [Fre17, MR04] for up to date discussions and positive results), one can propose a general conjecture for the values of the BCOV invariants of some Calabi–Yau varieties with complex multiplication. For this to be plausible, it seems however necessary to impose further conditions on the Hodge structure. Other recent examples of Calabi–Yau manifolds whose BCOV invariants should adopt a special form are given in [CdlOEvS19].

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DENNIS ERIKSSON, DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG

E-mail address: dener@chalmers.se

GERARD FREIXAS I MONTPLET, C.N.R.S. – INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE *E-mail address*: gerard.freixas@imj-prg.fr

CHRISTOPHE MOUROUGANE, INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES (IRMAR) E-mail address: christophe.mourougane@univ-rennes1.fr