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Highlights

- weak quadrupole S(2) 2-0 line in self-perturbed D2 is measured
- collisional line-shape effects and energy of this rovibrational transition are calculated
- the velocity-changing collisions are handled with the hard-sphere collisional kernel
- the experimental and theoretical pressure broadening and shift are consistent within 5
- we observe 3.4 sigma discrepancy between experimental and theoretical line position

ACCEPTED MANUSCRIPT

Accurate deuterium spectroscopy for fundamental studies

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Abstract

We present an accurate measurement of the weak quadrupole S(2) 2-0 line in self-perturbed D₂ and theoretical *ab initio* calculations of both collisional line-shape effects and energy of this rovibrational transition. The spectra were collected at the 247 - 984 Torr pressure range with a frequency-stabilized cavity ring-down spectrometer linked to an optical frequency comb (OFC) referenced to a primary time standard. Our line-shape modeling employed quantum calculations of molecular scattering (the pressure broadening and shift and their speed dependencies were calculated, while the complex frequency of optical velocity-changing collisions was fitted to experimental spectra). The velocity-changing collisions are handled with the hard-sphere collisional kernel. The experimental and theoretical pressure broadening and shift are consistent within 5% and 27%, respectively (the discrepancy for shift is 8% when referred not to the speed averaged value, which is close to zero, but to the range of variability of the speed-dependent shift). We use our high pressure measurement to determine the energy, ν_0 , of the S(2) 2-0 transition. The *ab initio* line-shape calculations allowed us to mitigate the expected collisional systematics reaching the 410 kHz accuracy of ν_0 . We report theoretical determination of ν_0 taking into account relativistic and QED corrections up to α^5 . Our estimation of the accuracy of the theoretical ν_0 is 1.3 MHz. We observe 3.4σ discrepancy between experimental and theoretical ν_0 .

Keywords: Molecular deuterium, Molecular collisions, Spectral line shapes, Absolute frequency measurements, Cavity ring-down spectroscopy, Quantum electrodynamics in molecules

1. Introduction

Molecular hydrogen in its ground electronic state, the simplest neutral chemically bound system, constitutes the most suitable platform for testing quantum electrodynamics (QED) for molecules and for searching for new physics beyond the standard model [1], such as new forces [2] or extra dimensions [3]. However, direct experimental studies on the rovibrational structure of H₂ are difficult to perform because of the absence of strong dipole coupling between the levels. Typical Doppler-free saturation spectroscopy is not applicable to weak quadrupole molecular transitions. Two independent experimental strategies have been advanced to reach the megahertz level of accuracy in the determination of the frequencies of rovibrational lines. The first one [4–8] is based on Doppler-free two-photon spectroscopy of electronic transitions in molecular beams. Two different vibrational levels in the ground electronic state were coupled to the same excited electronic level. This allowed the energy difference between the ($v = 1, j = 0$) and ($v = 0, j = 0$) ground electronic states to be determined with an accuracy of 2.8 MHz [7]. This approach has great potential for improvements because of recent progress in Ramsey-comb spectroscopic techniques [9]. The second strategy operates in the Doppler-limited regime. It takes advantage of ultra-high finesse cavities to directly measure

the weak quadrupole lines and has already achieved the kilohertz level of accuracy [10–12] typical for Doppler-free techniques. However, in the case of molecular hydrogen, the collisional line-shape effects are very pronounced and untypical [13–15]. As a result, the line position does not scale linearly with pressure [16]; this phenomenon usually is not taken into account [17]. To avoid such systematic errors, Mondelain et al. [18] measured the D₂ S(2) 2-0 line at very low pressure, where collisions are negligible, reaching an accuracy of 0.50 MHz.

In this article, we present experimental and theoretical studies on the shape of the D₂ S(2) 2-0 line, which allowed us to bring the previous measurements of the line position into the high-pressure range and reach (despite a twenty times shorter effective optical path) the same sub-megahertz level of accuracy as the Doppler-regime measurements [18]. The spectra were recorded with a frequency-stabilized cavity ring-down spectrometer linked to an optical frequency comb (OFC) referenced to a primary time standard. The experiment was carried out at room temperature and pressures varying from 247 to 984 Torr. In contrast to most of the works devoted to the analysis of the shapes of experimental molecular lines, where simple phenomenological models (such as hard- [19, 20] or soft-collision [21] models of the velocity-changing collision and quadratic or hypergeometric approximations of the speed dependence) are applied, we employ an approach originating from first principles

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[14, 15, 22, 23]. We performed fully quantum calculations of molecular scattering to obtain the pressure broadening and shift and their speed dependencies. The complex frequency of optical velocity-changing collisions was fitted to experimental spectra. The velocity-changing collisions are handled with a collisional kernel that originates from the hard-sphere approximation of the potential energy surface [14] (the resulting profile is called speed-dependent billiard-ball profile [24] with *ab initio* speed dependence). Our experimental and theoretical determinations of the pressure broadening are consistent within 5%. The relative discrepancy for the pressure shift is 27%. This value is, however, misleading because for this system the shift is large and strongly depends on absorber speed, but its speed-averaged value is close to zero. Therefore to provide better measure of the relative discrepancy one should refer the difference not to the speed-averaged value but to the range of the shift variability and then the relative discrepancy equals to 8%. The real part of the fitted frequency of the optical velocity-changing collisions is consistent with the frequency of the velocity-changing collisions calculated from kinetic theory. We use our measurement to determine the energy, ν_0 , of the S(2) 2-0 transition. The *ab initio* line-shape calculations allowed us to mitigate the expected collisional systematics (the line asymmetry originating from speed dependence of collisional shift) reaching the accuracy of 401 kHz.

We report theoretical determination of ν_0 taking into account relativistic and QED corrections up to α^5 . Our estimation of the accuracy of the theoretical ν_0 is 1.3 MHz. We observe 3.4σ discrepancy between experimental and theoretical ν_0 . The reason for this discrepancy is not known.

2. *Ab initio* line-shape modeling

2.1. Velocity distribution of an optical coherence at the stationary state

The shapes of molecular lines can be modeled by determining the velocity distribution of an optical coherence associated with the considered transition from the transport-relaxation equation [22, 23, 25–27]

$$f_m(\vec{v}) = -i(\omega - \omega_0 - \vec{k} \cdot \vec{v})f(\omega, \vec{v}) - \hat{S}f(\omega, \vec{v}), \quad (1)$$

where ω and ω_0 are the angular frequency of the electromagnetic radiation and the unperturbed angular frequency of the molecular transition. In the experimental data analysis, see Sec. 3, we use $\nu = \omega/(2\pi)$ and $\nu_0 = \omega_0/(2\pi)$. The $\vec{k} \cdot \vec{v}$ term is the Doppler shift, where \vec{v} is the velocity of an active molecule and \vec{k} is the wave vector. $f_m(\vec{v}) = (\sqrt{\pi}v_m)^{-3}e^{-(v/v_m)^2}$ is the Maxwell velocity distribution, where $v_m = \sqrt{2k_B T/m_1}$ is the most probable speed. k_B , T and m_1 are the Boltzmann constant, temperature and mass of the active molecule, respectively. \hat{S} is the collision operator describing the relaxation and dephasing of the optical coherence as well as its flow between different velocity classes. Since the D₂ rovibrational lines are very well separated (the rotational constant is exceptionally large, about 30 cm⁻¹), we restrict our discussion to the case of isolated lines, i.e., we neglect line mixing [28]. Having $f(\omega, \vec{v})$, which is a scalar function proportional to the

velocity distribution of the optical coherence, the line-shape function can be calculated as

$$I(\omega) = \frac{1}{\pi} \text{Re} \int f(\omega, \vec{v}) d^3\vec{v}. \quad (2)$$

The Maxwell distribution is factored out from the solution of Eq. (1), $f(\omega, \vec{v}) = f_m(\vec{v})h(\omega, \vec{v})$; hence, the equation takes the following form

$$1 = -i(\omega - \omega_0 - \vec{k} \cdot \vec{v})h(\omega, \vec{v}) - \hat{S}^f h(\omega, \vec{v}), \quad (3)$$

where \hat{S}^f is defined as $\hat{S}^f f_m(\vec{v})h(\omega, \vec{v}) = f_m(\vec{v})\hat{S}^f h(\omega, \vec{v})$. In the general case, Eq. (3) can be solved by decomposing functions and operators in some basis and hence converting Eq. (3), which is a multidimensional integral Boltzmann-like equation, into a set of algebraic equations. If we introduce an orthonormal basis, $\{\phi_s(\vec{v})\}$, and define a scalar product, $\langle \phi_{s'} | \phi_s \rangle = \int d^3\vec{v} f_m(\vec{v}) \phi_{s'}(\vec{v}) \phi_s(\vec{v})$, then Eq. (3) can be written as [27]

$$\underline{\mathbf{b}} = (-i(\omega - \omega_0)\mathbf{1} + i\mathbf{K} - \mathbf{S}^f)\underline{\mathbf{c}}(\omega), \quad (4)$$

where column $\underline{\mathbf{c}}(\omega)$ is $c_s(\omega) = \langle h(\omega, \vec{v}) | \phi_s(\vec{v}) \rangle$, matrix \mathbf{K} is $[\mathbf{K}]_{s',s} = \langle \phi_{s'} | \vec{k} \cdot \vec{v} | \phi_s \rangle$, $\mathbf{1}$ is the identity matrix, $\underline{\mathbf{b}}$ is a column defined as $[\underline{\mathbf{b}}]_s = \delta_{s,0}$ ($\delta_{s',s}$ is the Kronecker delta), and matrix \mathbf{S}^f is defined as $[\mathbf{S}^f]_{s',s} = \langle \phi_{s'} | \hat{S}^f | \phi_s \rangle$. In practice, the infinite system of coupled linear equations (4) is reduced to a system of at most a few thousand equations. If the first element of the basis is a constant function, i.e., $\phi_0(\vec{v}) = 1$, then the line-shape function, Eq. (2), can be written as [27]

$$I(\omega) = \frac{1}{\pi} \text{Re} \langle \phi_0(\vec{v}) | h(\omega, \vec{v}) \rangle = \frac{1}{\pi} \text{Re} [c_0(\omega)]. \quad (5)$$

In this work, we use the Burnett functions basis defined as

$$\psi_{nlm}(\vec{v}/v_m) = \sqrt{2\pi^{3/2}n!/\Gamma(n+l+3/2)} \cdot (v/v_m)^l L_n^{l+1/2}((v/v_m)^2) Y_{lm}(\theta, \phi), \quad (6)$$

where $L_n^{l+1/2}((v/v_m)^2)$ is an associated Laguerre polynomial, $Y_{lm}(\theta, \phi)$ is the usual spherical harmonic function, and $\{v, \theta, \phi\}$ are spherical coordinates of \vec{v} .

2.2. The collisional operator \hat{S}^f

The collisional operator, \hat{S}^f , can be arbitrarily divided into two components [29]:

$$\hat{S}^f = \hat{S}_D^f + \hat{S}_{\text{VCD}}^f, \quad (7)$$

where the \hat{S}_D^f operator describes the relaxation of the internal motion of the molecule, i.e., the damping and dephasing of the optical coherence. The \hat{S}_{VCD}^f operator describes the remaining part of \hat{S}^f , i.e., the flows of the optical coherence between different velocity classes affected by its simultaneous damping and dephasing.

In the idealized case when the changes in the molecule velocity are completely independent from (uncorrelated with)

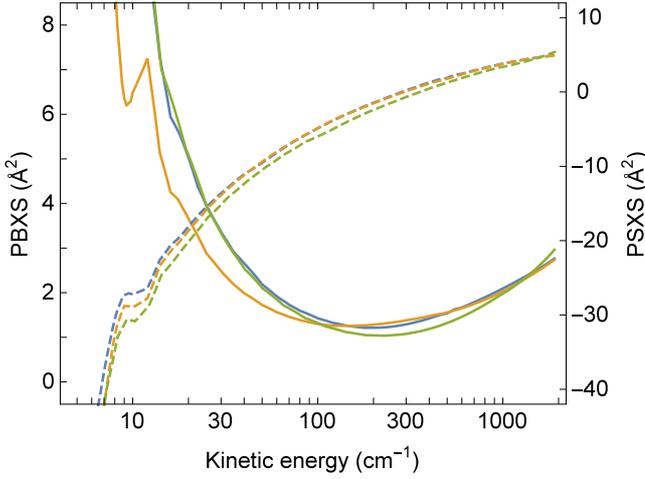


Figure 1: Generalized spectroscopic cross sections for the S(2) 2-0 transition in D₂ as a function of collisional kinetic energy. Pressure-broadening cross sections (PBXS) and pressure-shift cross sections (PSXS) are shown as solid and dashed lines, respectively. For readability, we show the results for three rotational states of perturbing molecules (out of six used in our calculations, i.e., $j_2 = 0$ to 5). The blue, yellow and green colors correspond to $j_2 = 0, 1$ and 4, respectively.

the damping and dephasing, then \hat{S}_{VCD}^f is equal to the usual Boltzmann collisional operator \hat{S}_{VC}^f [25, 30, 31]

$$\hat{S}_{\text{VC}}^f h(\omega, \vec{v}) = n_2 \int d^3 \vec{v}_2 \int d\tilde{\Omega} \left(\frac{d\sigma}{d\tilde{\Omega}} \right) |\vec{v} - \vec{v}_2| \cdot f_{m_2}(\vec{v}_2) (h(\omega, \vec{v}') - h(\omega, \vec{v})), \quad (8)$$

where n_2 is the concentration of perturbers, $\tilde{\Omega}$ is the scattering solid angle, $d\sigma/d\tilde{\Omega}$ is the differential elastic cross section, \vec{v}' is the velocity of an absorber after collision, \vec{v}_2 is the velocity of a perturber, and $f_{m_2}(\vec{v}_2)$ is the velocity distribution of the perturbers. The \hat{S}_{VC}^f operator can also be expressed in terms of the collision kernel $f(\vec{v} \leftarrow \vec{v}')$ describing probability rates of velocity changes from \vec{v}' to \vec{v} :

$$\hat{S}_{\text{VC}}^f h(\omega, \vec{v}) = \int d^3 \vec{v}' \left[\frac{f_m(\vec{v}')}{f_m(\vec{v})} f(\vec{v} \leftarrow \vec{v}') h(\omega, \vec{v}') - f(\vec{v}' \leftarrow \vec{v}) h(\omega, \vec{v}) \right]. \quad (9)$$

To quantify the frequency of the velocity changes, we also define a normalized form, \hat{M} , of this operator

$$\hat{S}_{\text{VC}}^f = \nu_{\text{VC}} \hat{M}^f, \quad (10)$$

where ν_{VC} is the effective frequency of the velocity-changing collisions, which can be calculated as

$$\nu_{\text{VC}} = \frac{v_m^2}{2D}, \quad (11)$$

where D is the mass diffusion coefficient.

Equations (8) to (11) are valid when there is no correlation between the velocity-changing and damping/dephasing collisions, which is not the case for realistic systems. The correlations are

accounted for in the generalized Hess method (GHM) [32–34]. Within the GHM framework, the \hat{S}_{D}^f operator is simply a complex number, $-(\Gamma_0 + i\Delta_0)$, where Γ_0 and Δ_0 are speed-averaged collisional broadening and shift, respectively. The GHM allows the \hat{S}_{VCD}^f operator to be identified with $\nu_{\text{opt}} \hat{M}_{\text{HC}}^f$, where \hat{M}_{HC}^f is the normalized Boltzmann collisional operator, see Eqs. (9) and (10), for the case of the simple hard-collision kernel

$$f(\vec{v} \leftarrow \vec{v}') = \nu_{\text{opt}} f_m(\vec{v}). \quad (12)$$

Therefore, the GHM allows us to write the full collision operator \hat{S}^f as

$$\hat{S}^f = -\Gamma_0 - i\Delta_0 + \nu_{\text{opt}} \hat{M}_{\text{HC}}^f. \quad (13)$$

The ν_{opt} parameter is called frequency of optical velocity-changing collisions; it has to be emphasized that it is a complex number. The real part of it, ν_{opt}^r , is smaller than ν_{VC} . The reason for this is that some part of the optical coherence, which could flow to different velocity classes, is damped or dephased. The nonzero imaginary part, ν_{opt}^i , results from the fact that during the velocity-changing collision, the phase of the optical coherence is changed as well. Within GHM, the Γ_0 , Δ_0 and ν_{opt} parameters are calculated from first principles as

$$\Gamma_0 + i\Delta_0 = \frac{1}{2\pi c} \frac{p}{k_B T} \sum_{j_2} p_{j_2} \omega_0^{00}(q, j_2), \quad (14)$$

$$\nu_{\text{opt}} = \frac{1}{2\pi c} \frac{p}{k_B T} M_2 \sum_{j_2} p_{j_2} \left[\frac{2}{3} \omega_1^{11}(q, j_2) - \omega_0^{00}(q, j_2) \right], \quad (15)$$

where the collision integral $\omega_\lambda^{s,s'}(q, j_2)$ is given by the following expression [35, 36]:

$$\omega_\lambda^{s,s'}(q, j_2) = \langle v_r \rangle \int_0^\infty dx x^{(s+s'+2)/2} e^{-x} \sigma_\lambda^q(v_i v_f v_j j_2; E_{kin} = x k_B T), \quad (16)$$

where $\langle v_r \rangle = \sqrt{8k_B T / \pi \mu}$ and μ is the reduced mass of the colliding partners. q is the tensor rank of the spectral transition operator (for the quadrupole line considered here $q = 2$). T , k_B and p are the temperature, Boltzmann constant and pressure, respectively. The $1/(2\pi)$ factor appears in Eqs. (14) and (15) because the horizontal axis of the spectrum is expressed in terms of frequency, ν , and not angular frequency, ω , as it is in Eq. (1). The $1/c$ factor converts the frequency unit into wavenumbers. $M_2 = m_2/(m_1 + m_2)$, where m_1 and m_2 are the masses of the active and perturbing molecules. In the self-perturbed case, as it is in this paper, $M_2 = 1/2$. p_{j_2} is the population of the j_2 state of the perturber molecule at temperature T (j_2 is the rotational quantum number of the perturbing molecule before the collision)

$$p_{j_2} = w_{j_2} (2j_2 + 1) e^{-E_{j_2}/(k_B T)} / Z(T), \quad (17)$$

where $Z(T)$ is the corresponding partition function

$$Z(T) = \sum_{j_2} w_{j_2} (2j_2 + 1) e^{-E_{j_2}/(k_B T)}. \quad (18)$$

The w_{j_2} weight arises from the symmetry condition for the total wavefunction of the D_2 molecule (the deuterium atom is a boson) and the degeneracy in the total nuclear spin I . For even j_2 , the total nuclear spin can be $I = 0$ or 2 ; hence, $w_{j_2} = 6$ for ortho-states. For odd j_2 , the total nuclear spin is $I = 1$; hence, $w_{j_2} = 3$ for para states. E_{j_2} is the energy of the j_2 state ($E_{j_2=0} = 0$). $\sigma_\lambda^q(v_i j_i v_f j_f j_2; E_{kin})$ is a generalized spectroscopic cross section, where $v_i j_i$ and $v_f j_f$ are vibrational and rotational quantum numbers for the initial and final spectroscopic states (for the 2-0 S(2) line $v_i = 0, j_i = 2, v_f = 2$ and $j_f = 4$). E_{kin} is the initial relative center-of-mass kinetic energy. The $\sigma_\lambda^q(v_i j_i v_f j_f j_2; E_{kin})$ cross sections are calculated from the scattering S-matrices. The explicit expression for $\sigma_\lambda^q(v_i j_i v_f j_f j_2; E_{kin})$ is given in Refs. [34] and [35]; see Eq. (34) in Ref. [34] and Eq. (2) in Ref. [35]. Note that our $\sigma_\lambda^q(v_i j_i v_f j_f j_2; E_{kin})$ has dimensions of area, while that in Ref. [34] is dimensionless. The conversion factor is π/k^2 , where $k^2 = 2\mu E_{kin}/\hbar^2$. Note that σ_0^q is a standard generalized spectroscopic cross section [37–39]. In Fig. 1, we present illustrative examples of calculated values of $\sigma_0^q(v_i = 0, j_i = 2, v_f = 2, j_f = 4, j_2; E_{kin})$ for $j_2 = 0, 1$ and 4 . We refer to the real and imaginary parts of σ_0^q as the pressure-broadening cross-section (PBXS) and the pressure-shifting cross-section (PSXS), respectively. We truncated the summation over j_2 at 5. At the considered temperature, $T = 294.9$ K, the cumulated population of the $j_2 > 5$ states is 0.6 % of the total population (the j_2 dependence is weak, and therefore, the corresponding error of the cross sections will be much smaller). Dynamical calculations were performed on Hinde’s PES [40].

and the S-matrices were determined by solving the close-coupling equations using the MOLSCAT code [41].

The hard-collision model is insufficient for a proper description of the velocity-changing collisions in the modeling of the shapes of molecular resonances, in particular, the shapes of molecular hydrogen lines [15, 16, 42–44]. The phenomenological hard-collision kernel, Eq. (12), obeys the basic thermodynamic requirements (at equilibrium, the velocity distribution converges to the Maxwellian distribution and the detailed balance relation is satisfied), but it is not based on the interaction potential. The physical meaning of the hard-collision kernel, Eq. (12), is that the velocity of the molecule is completely thermalized after each collision regardless of the pre-collisional velocity. The line-shape effects are physically better described when the VC collisions are obtained from the interaction potential of the colliding pair [14]. In panel (b) in Fig. 2, we show the isotropic part of the D_2 - D_2 potential (black line) and its hard-sphere approximation (blue line). As a reference, we also show the Maxwell-Boltzmann collision energy distribution in panel (a). It has been demonstrated, by a direct comparison with the *ab initio* classical molecular dynamics simulations [42], that the hard-sphere approximation very well describes the dynamics of the velocity-changing collisions for the case of self-perturbed hydrogen [14]. In contrast to the HC kernel, the hard-sphere kernel (also called the billiard-ball kernel, and hence, we refer to it as the BB kernel) depends on the velocity before the collision and the scattering angle θ [14, 45]

$$f_{BB}(\vec{v} \leftarrow \vec{v}') = v^{(0)} \frac{1}{v_m^2} \frac{3}{32\pi} \frac{\sqrt{1+\alpha}(1+\alpha)^2}{\alpha^2} \frac{1}{\sqrt{v^2 - 2vv' \cos \theta + v'^2}} \times \exp\left(-\frac{(1-\alpha)^2 v'^2}{4\alpha} \frac{1}{v_m^2} - \frac{(1+\alpha)^2 v^2}{4\alpha} \frac{1}{v_m^2} - \frac{(\alpha+1)(\alpha-1)vv'}{2\alpha} \frac{\cos \theta}{v_m^2} + \frac{\alpha v^2 v'^2 \sin^2 \theta}{v_m^2 (v^2 - 2vv' \cos \theta + v'^2)}\right), \quad (19)$$

where

$$v^{(0)} = (8/3)\sqrt{\pi}(\alpha^{-1} + 1)^{-1/2} v_m n_2 d^2 \quad (20)$$

is the first-order effective frequency of velocity-changing collisions [24, 46], α is the perturber-to-active-molecule mass ratio, d is the mean hard-sphere diameter of the colliding particles, and n_2 is the density number of perturbers. The relation with frequency of the velocity-changing collisions, ν_{VC} , is

$$\nu_{VC} = v^{(0)}/f_D, \quad (21)$$

where the α -dependent coefficient, f_D , can be determined from Eq. (19); see Refs. [14, 24, 46]. For the self-perturbed D_2 , $\alpha = 1$, $f_D \approx 1.01895$. In Fig. 2, panels (c) and (d), we compare the HC and BB kernels. It is clearly observed that the simple phenomenological HC kernel does not account for the angle and initial velocity dependencies and can differ from the real kernel by more than one order of magnitude. These unphysical

properties of the HC kernel are clearly observed in experimental spectra analysis as an inability to properly reproduce the shapes of molecular hydrogen rovibrational lines in a wide range of pressures [15, 16, 42–44]. It should be emphasized that the BB kernel also correctly handles the strong mass dependence of the kernel, whereas the HC kernel does not even depend on α . We denote the normalized operator of the velocity-changing collisions arising from the BB kernel as \hat{M}_{BB}^f . The direct expression for the decomposition of \hat{M}_{BB}^f in the Burnett basis is given in Refs. [24, 46].

For the case of molecular hydrogen (and all of its isotopologues), the speed dependence of the collisional broadening, Γ , and shift, Δ , is very pronounced and cannot be ignored in the experimental spectrum analysis. Therefore, the simple expression for Γ_0 and Δ_0 , Eq. (14), has to be replaced with its speed-dependent counterpart. Introducing the conditional probability, f_{m_2} , of having a relative speed v_r when the active

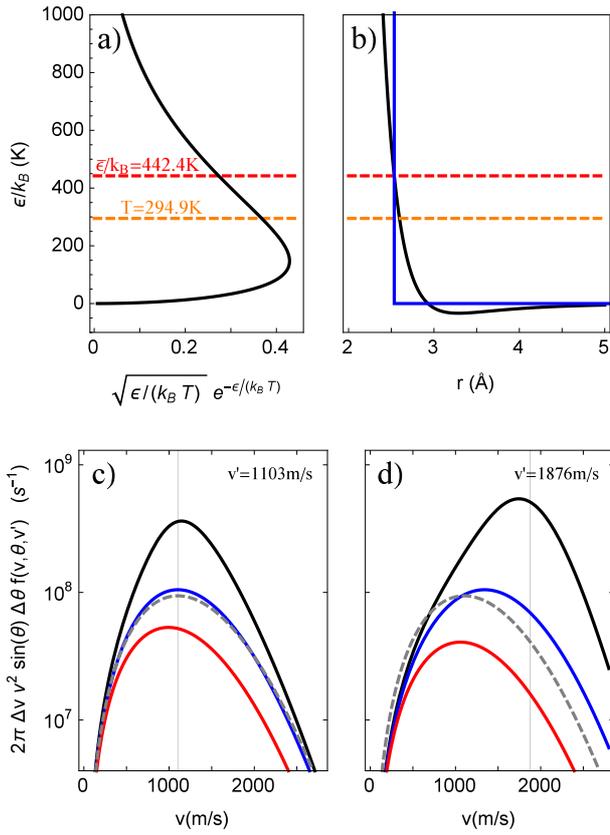


Figure 2: (a) Maxwell-Boltzmann collisional energy distribution at $T = 294.9$ K. (b) Isotropic part of the D_2 - D_2 potential. The blue line is the hard-sphere approximation whose diameter is set by choosing the intersection point at the mean collision energy, i.e., $\bar{\epsilon}/k_B = 442.4$ K. (c) and (d) Collision kernel for the D_2 - D_2 hard-sphere approximation at $T = 294.9$ K and $n_{H_2} = 1$ amg. The black, blue, and red colors correspond to scattering angles $\theta = 36^\circ$, 90° and 144° , respectively. Charts (c) and (d) were plotted for initial speeds $v' = 1563$ m/s and 2657 m/s. For comparison, the hard-collision kernel is plotted as a gray dashed line. The θ angle element $\sin\theta\Delta\theta = \Delta(-\cos\theta) = 0.2$, and the speed element $\Delta v = 200$ m/s.

molecule speed is v at the temperature T , the speed dependent collisional width and shift reads [47–49]:

$$\Gamma(v) + i\Delta(v) = \frac{1}{2\pi c} \frac{p}{k_B T} \sum_{j_2} p_{j_2} \int d^3\vec{v}_r f_{m_2}(\vec{v} + \vec{v}_r) v_r \sigma_0^q(v_i j_i v_f j_f j_2; v_r). \quad (22)$$

The integrals over the angle coordinates of \vec{v}_r can be performed analytically [47–49]:

$$\Gamma(v) + i\Delta(v) = \frac{1}{2\pi c} \frac{p}{k_B T} \sum_{j_2} p_{j_2} \frac{2}{\sqrt{\pi} \bar{v}_p v} \int_0^\infty dv_r v_r^2 e^{-\frac{v^2 + v_r^2}{\bar{v}_p^2}} \cdot \sinh\left(\frac{2vv_r}{\bar{v}_p^2}\right) \sigma_0^q(v_i j_i v_f j_f j_2; v_r), \quad (23)$$

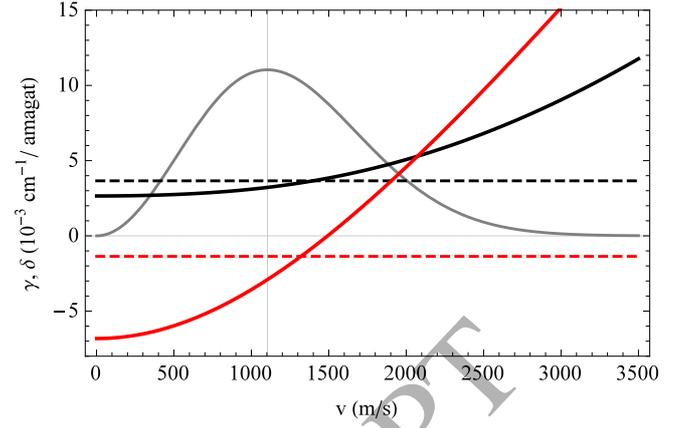


Figure 3: *Ab initio* speed dependences of the broadening, γ (black solid), and shift, δ (red solid), parameters for the deuterium S(2) 2-0 line at 294.9 K. These parameters are determined from fully quantum scattering calculations separately for each rotational state of perturbing D_2 and averaged over a proper nuclear spin statistic. As a reference, the gray curve shows the corresponding Maxwellian speed distribution (arbitrary units). The black and red dashed lines are the speed-averaged collisional broadening, γ_0 , and shift, δ_0 , respectively.

where \bar{v}_p is the most probable speed of the perturber distribution (in the self-perturbed case, $\bar{v}_p = v_m$). In Fig. 3, we show our *ab initio* $\Gamma(v)$ and $\Delta(v)$ for the 2-0 S(2) line. A direct expression for the decomposition of the $\Gamma(v)$ and $\Delta(v)$ functions in the Burnett basis is given in Ref. [24].

To account for the unphysical properties of the HC kernel and the speed dependence of Γ and Δ , following Refs. [24, 29, 50], we extend Eq. (13) to

$$\hat{S}^f = -\Gamma(v) - i\Delta(v) + v_{\text{opt}} \hat{M}_{\text{BB}}^f. \quad (24)$$

Depending on the pressure range, the transport-relaxation equation, Eq. (3), with the collisional operator given by Eq. (24) is solved either with a simple diagonalization method (at high pressures) [50, 51] or with an iterative method [52] in the full pressure range. The line shape originating from this approach, Eqs. (4), (5), (15), (23) and (24), is called the speed-dependent billiard-ball profile [24] with *ab initio* speed dependence and *ab initio* correlations between decoherence and velocity changes ($\text{SD}_{\text{ai}}\text{BBP}$).

3. Experimental spectra analysis

In this section, the details of the analysis of our experimental spectra are presented. We discuss the importance of employing a proper description of collisions in order to accurately describe the molecular line shapes. We also discuss the difficulties related to some technical issues such as the presence of etalons and the systematic errors resulting from numerical correlations between some parameters of the model.

3.1. Experimental setup

We measured the S(2) line from the 2-0 band of D_2 at four pressures (247.2, 471.3, 743.8 and 984.4 Torr) and at a temperature of 294.9 K. We used a deuterium sample having a

purity of 99.96%. The spectra were collected with a frequency-stabilized cavity ring-down spectrometer (FS-CRDS) linked to an optical frequency comb (OFC) [53]. The OFC was referenced to a primary time standard, the UTC(AOS) (Coordinated Universal Time from the Astro-Geodynamic Observatory in Borowiec, Poland). The length of the cavity was 74 cm, which corresponds to the free spectral range of 204 MHz. The cavity finesse was approximately 40000. The experimental setup is described in details in Ref. [54]. The collected spectra are shown in Fig. 4; see the red dots in the top panel. To achieve high signal-to-noise ratio (8500 at the highest pressure), approximately 85 spectra were averaged at each pressure.

3.2. Simple pressure-by-pressure fits with Voigt and hard-collision profiles

The simplest approach to determine the unperturbed position of the line, ν_0 , is to fit the spectra independently, pressure by pressure, with some simple symmetric profile and use a linear extrapolation of the pressure-dependent line position, $\nu_p(p)$, to the zero-pressure limit. We tested this approach with two models: the Voigt profile (VP) and the hard-collision profile (HCP). The residuals are shown in Fig. 4, panels (a) and (b), and the retrieved values of the parameters are shown in Table 1, rows (a) and (b). For the case of the VP, the values of the line-shape parameters are completely unphysical since the VP does not account for Dicke-narrowing, which is exceptionally strong in the case of D_2 . The Dicke-narrowed shape is artificially reproduced by significant diminishing of Γ_D and enlarging of γ_0 ; compare Γ_D and γ_0 retrieved with the VP with the *ab initio* values, rows (a) and (f) in Table 1. In the case of the HCP, the Γ_D parameter was fitted and not constrained to the value determined from the temperature (otherwise, the shape of the VP does not fit the experimental spectra at all). The values of the line-shape parameters are more reasonable when the HCP is employed since it considers Dicke-narrowing (in this case, Γ_D is fixed to the value determined from temperature). Nevertheless, the fitted parameters remain far from the real values, compare with rows (c) and (f) in Table 1, and the residuals are almost as bad as in the case of the VP; see panel (b) in Fig. 4. The shapes of the two models, the VP and the HCP, are different, but since both of them are symmetric, the pressure-dependent line positions, ν_p , determined with them are almost the same; see Fig. 5. The shapes of the D_2 lines are asymmetric due to collisional effects, and the degree of this asymmetry depends on the pressure (the asymmetry vanishes at low pressures and saturates at some constant pressure-independent level in the high-pressure limit). Consequently, ν_p determined with a symmetric-profile fit does not scale linearly with pressure (see the top panel in Fig. 2 in Ref. [16]), which is usually neglected in the line-shape analysis. We show this nonlinear behavior in the bottom panel of Fig. 5. The linear extrapolation of ν_p to the zero-pressure limit leads to a systematic error in ν_0 at the level of $0.18 \cdot 10^{-3} \text{ cm}^{-1}$ or 5 MHz; see $\delta\nu_0$ for the VP and the HCP in Table 1.

It should be noted that the residuals from the HCP, shown in panel (b) in Fig. 4, would be much larger if the multi-spectrum fitting approach would be applied and a more physical behavior would be enforced (for instance, enforcing the linear pressure

dependence of at least one of the collisional parameters or fixing some of these parameters to their real values).

3.3. Ultimate multi-spectrum fit with *ab initio* line-shape model

We eliminate the above-mentioned systematic error in the determination of ν_0 by employing the *ab initio* line-shape model ($SD_{ai}BBP$; see Sec. 2) in the analysis of our experimental spectra. To avoid numerical correlations between some parameters of the line-shape model and hence systematic errors in the retrieval of their values (in particular, in the value of ν_0), we use the multi-spectrum fit approach, and we fix the speed dependence of the broadening and shift to the *ab initio* values. A detailed discussion on the influence of the numerical correlations is given in Sec. 3.4. To quantify the strength of the speed dependence of the broadening and shift and to have the ability to adjust its magnitude in the fitting routines (see Sec. 3.4) while preserving the *ab initio* shape of these functions (see Fig. 3), we introduce two parameters, γ_{SD} and δ_{SD} , defined by the following equations:

$$\gamma(\nu) = \gamma_0 + \gamma_{SD} b_\gamma(\nu), \quad (25)$$

$$\delta(\nu) = \delta_0 + \delta_{SD} b_\delta(\nu). \quad (26)$$

To ensure the uniqueness of the choice of the γ_{SD} and δ_{SD} parameters, an additional normalization-like condition has to be implied in the dimensionless functions $b_\gamma(\nu)$ and $b_\delta(\nu)$:

$$\left. \frac{d}{d\nu} b_\gamma(\nu) \right|_{\nu=\nu_m} = \left. \frac{d}{d\nu} b_\delta(\nu) \right|_{\nu=\nu_m} = \frac{2}{\nu_m}. \quad (27)$$

It should be noted that since γ_0 and δ_0 are speed-averaged values of $\gamma(\nu)$ and $\delta(\nu)$, the average of $b_\gamma(\nu)$ and $b_\delta(\nu)$ over the Maxwell velocity distribution is zero. Direct expressions for calculating γ_{SD} and δ_{SD} from *ab initio* $\gamma(\nu)$ and $\delta(\nu)$ functions are

$$\gamma_{SD} = \frac{\nu_m}{2} \left. \frac{d}{d\nu} \gamma(\nu) \right|_{\nu=\nu_m}, \quad (28)$$

$$\delta_{SD} = \frac{\nu_m}{2} \left. \frac{d}{d\nu} \delta(\nu) \right|_{\nu=\nu_m}. \quad (29)$$

Our best determination of the line position and the line-shape parameters, retrieved with $SD_{ai}BBP$, are given in row (c) in Table 1. The broadening parameter, γ_0 , agrees with the *ab initio* value within approximately 5%; compare with row (f) in Table 1. Much worse agreement, approximately 27%, is observed for the shift parameter δ_0 . The discrepancy is caused by the PES [40] inaccuracy (the line shift is substantially more sensitive to PES imperfections than the line broadening). A new H_2 - H_2 PES, which is under development [55], will allow us to reduce this discrepancy considerably. It has to be emphasized that for the determination of the line position, ν_0 , the inaccuracy of the *ab initio* line shift, δ_0 , is not that crucial because we fit the value of this parameter to experimental spectra. The critical part of our analysis is the *ab initio* value of δ_{SD} since we keep it fixed in the multi-spectrum fit procedure. This is one of the parameters determining the asymmetry of the line (see the discussion in

Table 1: Line position and line-shape parameters ($T = 294.9$ K) retrieved from fits to our experimental spectra with different approaches. ν_0 is the unperturbed line position, $\delta\nu_0$ is the deviation of ν_0 from our best determination, Γ_D is the Gaussian half width at half maximum (HWHM), γ_0 and δ_0 are the pressure broadening and shift parameters, γ_{SD} and δ_{SD} are parameters that quantify the strength of the broadening and shift speed dependence (see the text for details), and $\tilde{\nu}_{opt}^i$ and $\tilde{\nu}_{opt}^r$ are the real and imaginary parts of the frequency of the optical velocity-changing collisions (also called Dicke-narrowing parameter). Rows (a) - (e) present the values of the line-shape parameters retrieved from experimental spectra with different approaches. The corresponding residuals are shown in panels (a) - (e) in Fig. 4. Row (c) shows the parameters retrieved from our ultimate fit. The last three columns define the conditions of the fits. SD denotes the speed dependence of the collisional broadening and shift. The last row shows the theoretical *ab initio* values. The error bars in rows (a) to (e) are 1σ standard uncertainties retrieved from the fits' residuals and the covariance matrix. In particular, the uncertainty of ν_0 in row (c) is the same as the number in the first row of Table 2. The error bars in the last row are our estimations of the 1σ uncertainties of the theoretical calculations (see the text for details). The zero uncertainty means that it is smaller than one at the last reported digit. ν_0 , $\delta\nu_0$ and Γ_D are in 10^{-3} cm $^{-1}$, whereas all the other parameters are in 10^{-3} cm $^{-1}$ /atm.

Line-shape model	ν_0	$\delta\nu_0$	Γ_D	γ_0	δ_0	γ_{SD}	δ_{SD}	$\tilde{\nu}_{opt}^r$	$\tilde{\nu}_{opt}^i$	Etalons	SD	Fit type
a) VP	6241127.479(29)	-0.176	3.6(1.5)	12.0(3.5)	-1.93(3)	-	-	-	-	no	-	line-by-line
b) HCP	6241127.474(22)	-0.181	19.12(fixed)	1.96(53)	-1.89(3)	-	-	20.2(5)	-	no	-	line-by-line
c) SD_{all}BBP (ultimate fit)	6241127.655(4)	0.000	19.12(fixed)	3.57(0)	-1.75(0)	0.62(fixed)	3.41(fixed)	30.09(3)	-5.76(2)	yes	fixed	multi spect.
d) SD _{all} BBP (all param. fitted)	6241127.625(8)	-0.030	19.12(fixed)	4.06(1)	-1.91(1)	-1.27(8)	0.18(7)	34.9(2)	1.0(1)	yes	fitted	multi spect.
e) SD _{all} BBP (without etalons)	6241127.582(11)	-0.073	19.12(fixed)	3.51(1)	-1.74(1)	0.62(fixed)	3.41(fixed)	29.68(6)	-6.15(5)	no	fixed	multi spect.
f) <i>Ab initio</i>	6241127.515(37)	-0.140	19.12(1)	3.38(11)	-1.27(27)	0.62(2)	3.41(15)	30.7(2.3)	-	-	-	-

Sec. 3.4); hence, if not correctly handled, it may be responsible for the systematic error in the retrieved line position ν_0 . The systematic error of *ab initio* δ_{SD} is substantially smaller than the mentioned error of the *ab initio* δ_0 . One of the reasons is that the $\delta(\nu)$ function (red line in Fig. 3) is almost symmetrically distributed around zero shift, and consequently, the average value, i.e., δ_0 (the red dashed line in Fig. 3), is close to zero. The relative error of δ_0 is as large as 27 % because the reference value (i.e., the value of δ_0) is close to zero. If we refer the absolute difference between the *ab initio* and experimental δ_0 not to the value of δ_0 but to the range of variability of $\delta(\nu)$ then the relative error is much smaller. This is why we expect a substantially smaller relative error of δ_{SD} , i.e., the error of the slope of the $\delta(\nu)$ function. To give a reliable estimation of the uncertainty of our *ab initio* δ_{SD} , we take advantage of the recent measurements [56] of the temperature dependence of δ_0 for the same line. It should be noticed that both the speed and temperature dependencies of δ_0 have the same physical origin, i.e., the energy dependence of the spectroscopic cross-section. Therefore, one of them automatically gives information about the other. Zaborowski et al. [56] recently measured the temperature dependence $\delta_0(T) = \delta_0(T_{REF})(T_{REF}/T)^n$, with $\delta_0(T_{REF}) = -1.69(1) \cdot 10^{-3} \text{ cm}^{-1}/\text{atm}$, $n = 6.4(2)$ and $T_{REF} = 296 \text{ K}$, whereas our *ab initio* calculations gave $\delta_0(T_{REF}) = -1.23 \cdot 10^{-3} \text{ cm}^{-1}/\text{atm}$ and $n = 7.7$. Assuming a quadratic approximation of δ_{SD} (see Appendix A), the relation between δ_{SD} and n is given by a simple analytical formula: $\delta_{SD}(T) \approx \delta_2(T) = \delta_0(T)(1 - n)/3$ [44, 57, 58]. Using this expression, we estimated the uncertainty of our *ab initio* δ_{SD} as the difference between the *ab initio* and experimental δ_2 , which gives an absolute uncertainty $\sigma(\delta_{SD}) = 0.25 \cdot 10^{-3} \text{ cm}^{-1}/\text{atm}$ or relative uncertainty of 7 % (the value reported in Table 1 is expressed in terms of 1σ uncertainty). To estimate the influence of this uncertainty on the determined value of the line position ν_0 , we repeated the multi-spectrum fit procedure while changing the fixed value of δ_{SD} by a more conservative 10 %. The resulting uncertainty of ν_0 constitutes a dominant contribution to the uncertainty budget; see row (3) in Table 2. It has to be emphasized that a more accurate PES will allow the uncertainty of δ_{SD} to be reduced considerably - we have already demonstrated this [59] for the case of He-perturbed H_2 , for which a highly accurate PES is already available [60]. The theoretical value of ν_{opt}^r reported in Table 1 is a classical frequency of the velocity-changing collisions from the hard-sphere approximation, which is consistent (at room temperature) with the more realistic value calculated from a mass diffusion coefficient within approximately 3 % [44]. A larger contribution to the uncertainty originates from the fact that ν_{opt}^r may differ from the usual frequency of the velocity-changing collisions by as much as γ_0 . Therefore, we estimate the 1σ uncertainty of ν_{opt}^r at $2.3 \cdot 10^{-3} \text{ cm}^{-1}/\text{atm}$. One should note that this quantity serves here as a reference only and it does not influence the uncertainty of the ν_0 determination.

The fitted line area, within its statistical uncertainty, does not exhibit any deviations from linear scaling with pressure and the intercept value is consistent with zero; hence, the influence of the pressure gauge accuracy on the ν_0 determination is negligible. We estimated the influence of the temperature instability on ν_0 by examining the correlation between the temperature

Table 2: Estimated contributions to the standard uncertainty budget of our experimental determination of the frequency of the S(2) 2-0 transition in D_2 , $\nu_0 = 187\,104\,300.038(401) \text{ MHz}$.

Uncertainty source (type)	$u(\nu_0) / \text{kHz}$
1) Statistics, 1σ (A)	132
2) Optical frequency comb (A+B)	< 1
3) Line-shape analysis (B)	357
4) Instrumental systematic shift (B)	47
5) Relativistic asymmetry (B)	< 3
6) Pressure gauge nonlinearity (B)	< 1
7) Etalons (B)	59
8) Temperature instability (A+B)	100
Standard combined uncertainty	401

and pressure. Note that for small temperature deviations the temperature dependence of the pressure shift is linear, hence instead of checking the pressure dependence of the temperature variations of the line shift we can check the pressure dependence of temperature. The difference between the mean temperature and its extrapolation to zero pressure is 0.11 K. The temperature dependence of δ_0 , determined from our *ab initio* calculations and validated with experiment, is smaller than 1 MHz/K at 1 atm and 294.9 K, which allows us to estimate the temperature instability contribution as 100 kHz; see Table 2.

3.4. Numerical correlations between the parameters of the line-shape models and influence of the etalons

In the previous section, we noted that the main source of the uncertainty of our determination of ν_0 is the uncertainty of the fixed *ab initio* magnitude of the speed dependence (mainly, δ_{SD}). One may wonder whether this uncertainty can be reduced simply by fitting the γ_{SD} and δ_{SD} parameters (together with all the other line-shape parameters) to experimental spectra. In this section, we show that this straightforward approach is very problematic and does not provide good control over the systematic errors. The reason for this is a strong numerical correlation between the line-shape parameters, i.e., several line-shape parameters have very similar influences on the shape of the molecular line, and instead of retrieving their real values, the fitting algorithm reconstructs the shape of the line in a geometrical sense by adjusting unphysical values of the line-shape parameters. For us, particularly unwanted is a correlation between the line-shape parameters responsible for the asymmetry of the line, i.e., δ_{SD} and ν_{opt}^i , and the parameters effectively determining the position of the line, i.e., ν_0 and δ_0 . To some extent, they can be decorrelated by employing the multi-spectrum fit approach, i.e., fitting the spectra at all the pressures simultaneously and constraining a proper pressure dependence of the parameters at the same time [63, 64]. It turns out, however, that this is insufficient. In row (d) of Table 1, we show the results of the same analysis as in our ultimate fit, which can be observed in row (c), but also adjusting the parameters γ_{SD} and δ_{SD} . Clearly, the retrieved values of the line-shape parameters suffer from much larger systematic errors, and some of them are completely

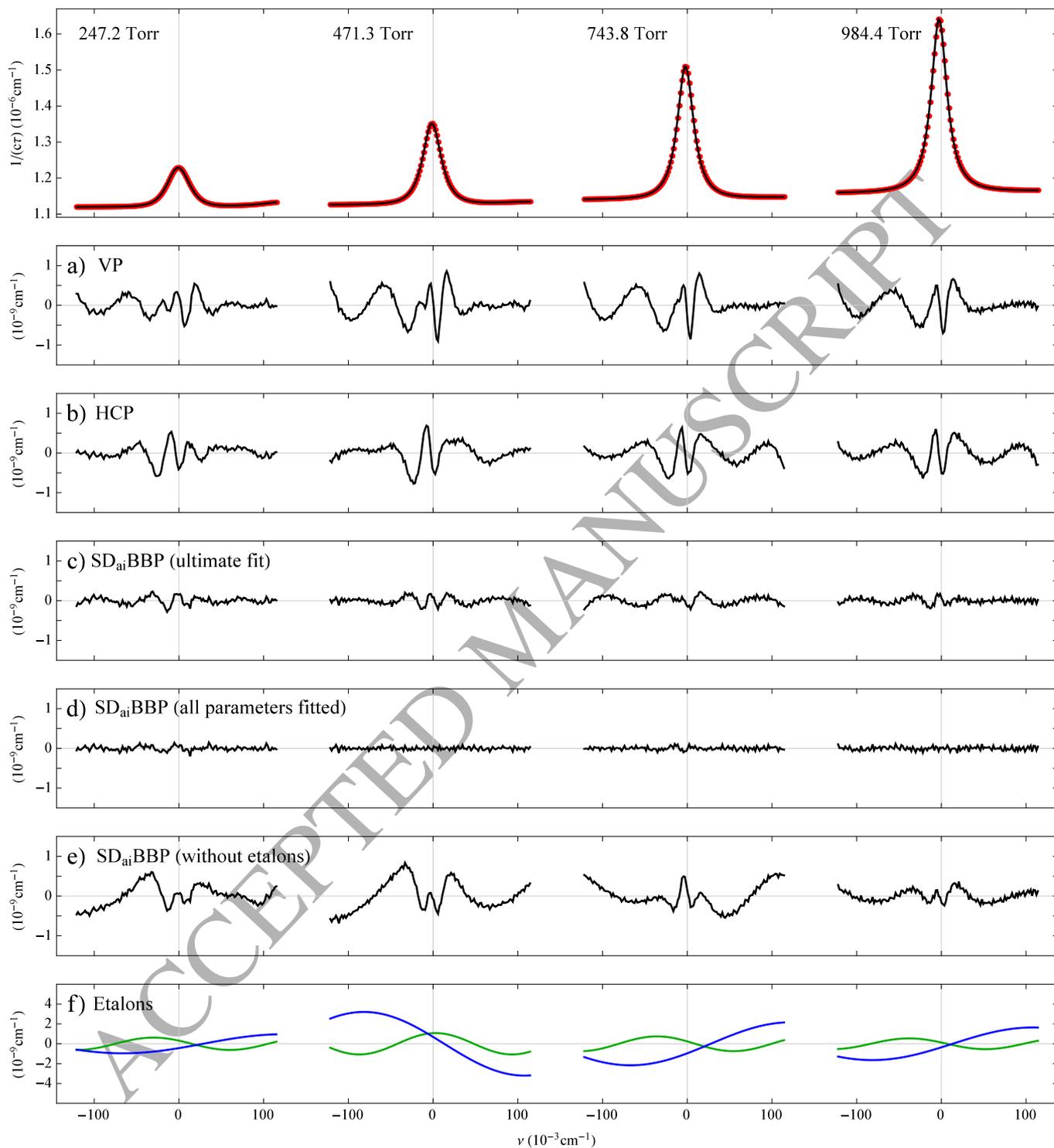


Figure 4: Rovibrational 2-0 S(2) line of D_2 recorded at 247.2, 471.3, 743.8 and 984.4 torr. Panels (a) to (e) show the residuals from fits obtained with different approaches. For additional details, see Table 1. The (a)-(e) denotation is the same for the above panels and the rows in Table 1. In panel (e), we show the two etalons retrieved from our ultimate multi-spectrum fits. The blue and green lines are the etalons with fixed periods of 5.4 and 11.4 GHz, respectively, whose phases and amplitudes were fitted.

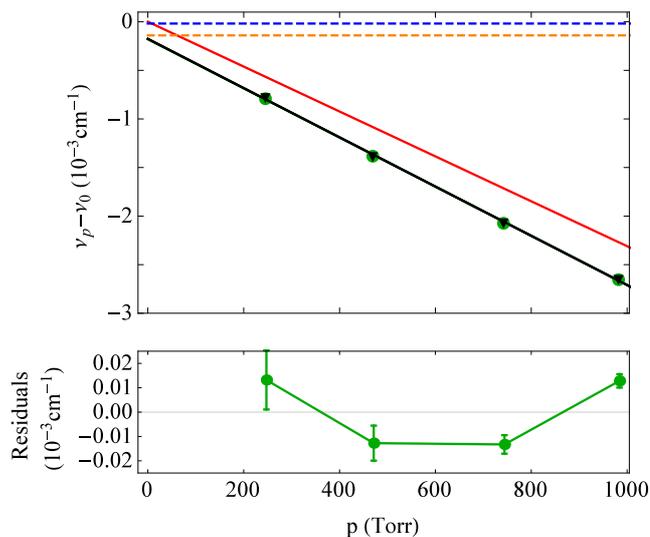


Figure 5: Pressure dependence of the collision-shifted line position, ν_p , for the pressure-by-pressure fits with the Voigt profile (black triangles) and the hard-collision profile (green circles). The black and green lines are linear fits to the Voigt and hard-collision profile positions, respectively (the green line is covered by the black one). The red line is the linear pressure shift, $\nu_0 + p\delta_0$, determined with the speed-dependent billiard-ball profile with *ab initio* speed dependence (here, we do not have points for every pressure since the multi-spectrum fit approach was employed), where $\nu_0 = 6241.127655 \text{ cm}^{-1}$ and $\delta_0 = -1.756 \cdot 10^{-3} \text{ cm}^{-1}/\text{atm}$. The dashed blue and orange horizontal lines indicate the experimental [18] and theoretical (this work) determinations of ν_0 . The lower panel shows the residuals from the linear fit to the green circles.

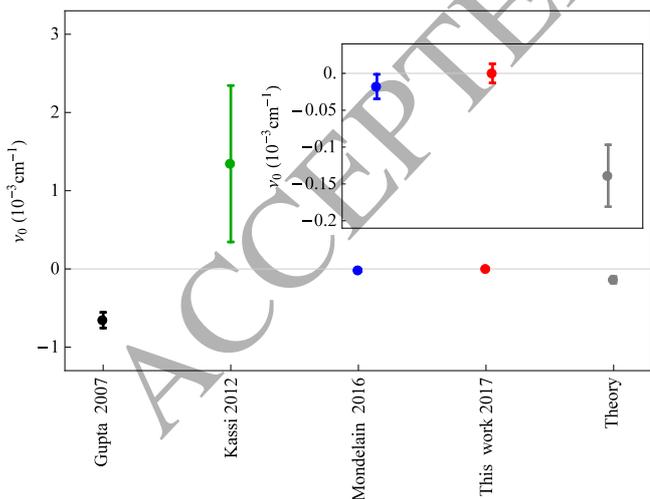


Figure 6: Comparison of our experimental (red dot) and theoretical (gray dot) determination of the energy of the S(2) 2-0 line in D_2 with the previous experimental results: Gupta et al. [61] (black dot), Kassi et al. [62] (green dot) and Mondelain et al. [18] (blue dot). The inset shows the last three points zoomed in on.

unphysical. For instance, the γ_{SD} parameter (responsible for profile narrowing) is strongly correlated with γ_0 and ν_{opt}^r because all these parameters effectively determine the width of the line (none of them influence the asymmetry of the line). A comparison of their values with the results in rows (c) and (f) in Table 1 reveals the unphysical flow of the contributions between them. Larger γ_0 and smaller γ_{SD} increase the profile width, which is compensated by a larger value of the Dicke-narrowing parameter. The retrieved value of γ_{SD} is negative. This is completely unphysical because it implies that, for a large range of D_2 speeds, the pressure broadening is negative, which does not make any physical sense. Similarly, the considerably overvalued ν_{opt}^r is certainly unphysical (the presence of the relaxations in the system makes the ν_{opt}^r parameter smaller than the frequency of the velocity-changing collisions, but there is no physical mechanism that could make it larger). The same problems occur in the case of the parameters responsible for the asymmetry of the line, i.e., δ_{SD} and ν_{opt}^i . The fitted value of δ_{SD} is nearly zero, which strongly contradicts both our *ab initio* calculations and the recent experimental temperature dependence of δ_0 [56]. The unphysical asymmetry of the line shape indispensably results in an unphysical shift in the retrieved zero-pressure line position ν_0 (at the 1 MHz level in this case). Concluding, when all the line-shape parameters are fitted, due to the strong numerical correlations, the retrieved values are completely unphysical. Within this approach, we do not have a tool to estimate the systematic error of ν_0 ; therefore, this approach is useless from the point of view of the ultra-accurate spectroscopy of the rovibrational structure in weakly interacting molecules. Finally, one should also note that when all the line-shape parameters are fitted, the residuals are obviously smaller; see panel (d) in Fig. 4. However, despite that, the uncertainties in the retrieved parameters are considerably larger (including the ν_0 parameter); see Table 1. This is a direct manifestation of the strong numerical correlations (the residuals are smaller, but the correlations are effectively present in the covariance matrix).

In our experimental setup, two weak etalons [65] are present with periods of 5.4 GHz and 11.4 GHz; see the green and blue lines in panel (f) in Fig. 4 (they originate from unwanted reflections from the vacuum chamber windows and are manifested in the spectra as an additional sinusoidal contribution). These periods well correspond to the distances between the highly reflective surfaces of the cavity mirrors and the surfaces of the vacuum chamber windows, which are approximately 3 and 1.5 cm, respectively. We have been measuring the periods of the two etalons with an empty cavity in a wide frequency range. These periods are stable over the course of at least six months. To avoid introducing additional fitted parameters numerically correlated with ν_0 , the periods of the etalons were fixed, and only their phases and amplitudes were fitted. The strong side of our approach is that we enforce the correct physical behavior of the line shape as a function of pressure by not only employing the multi-spectrum fit but also fixing the experimentally validated [56] *ab initio* speed dependence. This allows us to avoid situations wherein some unphysical combination of the line-shape parameters and etalons phases and amplitudes is fitted. To illustrate the necessity of employing the two etalons in our analysis, we show in panel (e) in Fig. 4 the residuals from the same fit as in our ultimate

approach but without fitting the etalons; the retrieved values of the line-shape parameters are given in row (e) in Table 1. The fitted line-shape model was slightly perturbed to compensate the lack of the etalons in the spectrum model. This effectively leads to a systematic error in the ν_0 determination at the level of 2 MHz. In our ultimate fit, row (c) in Table 1, we estimated the etalons' contribution to the uncertainty budget to be 59 kHz by varying the fixed values of their periods by a conservative amount of 10 %; see row (7) in Table 2. It has to be emphasized that in future experiments aiming at kilohertz accuracy of ν_0 , the contribution of the etalons does not constitute a fundamental limitation because their influence can be considerably reduced in a few independent ways. For instance, the vacuum chamber windows can be covered with antireflection coatings and wedge mounted. The residual etalon structure can be better decorrelated from the line-shape structure by measuring the spectrum across a wider range (including very far wings where etalons dominates).

3.5. Experimental determination fo the line position

The value and uncertainty budget of the determined line position are given in Table 2. Figure 6 shows a comparison with the previous experimental determinations of ν_0 . Our result agrees with the value recently reported by the group from Grenoble [18], which has only slightly larger uncertainty. The dominant component of the uncertainty budget is the inaccuracy in the *ab initio* calculations of the speed dependence of the collisional shift. It determines the asymmetry of the line; hence, it can be responsible for the systematic error in the retrieved line position ν_0 . The total combined standard uncertainty of the measurement is 401 kHz. Despite the twenty-times-shorter effective optical path, we achieved the same sub-megahertz level of accuracy as the previous Doppler-regime measurements [18] ($\sigma = 500$ kHz in Ref. [18]).

4. *Ab initio* calculations of rovibrational splitting

In this section we describe our theoretical determination of ν_0 , which takes into account relativistic and QED corrections up to α^5 (α is the fine-structure constant). The experimental value of the energy of the S(2) 2-0 transition in D₂ reported here differs from our theoretical determination by 3.4σ ; see Table 3. The reason for this discrepancy is not known. We suppose that it is most probably caused by underestimation of the combined relativistic nonadiabatic effects and not by the loss of numerical precision. The nonrelativistic energy is obtained here via a fully nonadiabatic approach with explicitly correlated exponential functions using octuple precision arithmetics [66]. The relativistic corrections $E^{(2)}$ and $E^{(4)}$, taken from [67, 68], were obtained using explicitly correlated Gaussian functions that exactly satisfy the interelectronic cusp condition, thus ensuring high numerical precision. Only the QED correction $E^{(3)}$ [69], calculated using the standard explicitly corrected Gaussian functions, exhibits some numerical uncertainty of relative order 10^{-3} . The higher order correction $E^{(5)}$ is only estimated using the known atomic hydrogen results and assuming that it scales

Table 3: Contributions to the theoretical determination of the energy of the S(2) 2-0 transition in D₂. $E^{(2)}$, $E^{(3)}$, $E^{(4)}$, and $E^{(5)}$ are calculated in the Born-Oppenheimer (BO) approximation; therefore, they contain the uncertainty due to nonadiabatic corrections, denoted by (na) , while the other uncertainty corresponds to approximate numerical evaluation. Lack of the specified uncertainty means that it is negligible. E_{FS} is the finite nuclear size correction. In the bottom part of the table, we compare the recent experimental determination [18] with our measurement. The last row is the combined value of these two experimental energies.

Contribution	cm ⁻¹
$E^{(0)}$ (nonrelativistic)	6241.120920(1)
$E^{(2)}$ (α^2 relativistic)	0.040057(20) _{na}
$E^{(3)}$ (α^3 QED)	-0.03315(3)(2) _{na}
$E^{(4)}$ (α^4 QED)	-0.000299
$E^{(5)}$ (α^5 QED)	0.000019(10)
E_{FS}	-0.000032
Total theor.	6241.127515(42)
Expt. ^a	6241.127637(17)
Expt. ^b	6241.127655(13)
Combined expt.	6241.127647(11)

^aRef. [18]

^bThis work

proportionally to the square of the electron wave function at the nuclei. However, all the relativistic contributions have been calculated within the Born-Oppenheimer approximation and thus neglect the combined relativistic nonadiabatic corrections. These corrections usually scale with the ratio of the electron to the reduced nuclear mass, resulting in a 5×10^{-4} factor. For the considered here S(2) line, it gives a quite small uncertainty of 2×10^{-5} . The problem is that the leading relativistic correction $E^{(2)}$ is anomalously small; as a result, this reduced mass scaling may not be valid in this particular case. Indeed we observed in Ref. [68] similar discrepancies for other transitions in H₂ and D₂. In conclusion, the present discrepancy is most likely caused by the underestimated combined nonadiabatic-relativistic correction.

5. Conclusion

In this article we presented accurate measurements of the weak quadrupole S(2) 2-0 line in self-perturbed D₂ in a pressure range from 247 - 984 Torr. We performed a detailed analysis of the collisional line-shape effects, which originates from fully quantum calculations of D₂-D₂ scattering. The velocity-changing collisions are handled with the collisional kernel based on hard-sphere approximation of the potential. The experimental and theoretical pressure broadening and shift are consistent within 5% and 8%, respectively (the discrepancy for the shift is referred not to the speed averaged value, which is close to zero, but to the range of variability of the speed-dependent shift). We use our high pressure measurement to determine the energy, ν_0 , of the S(2) 2-0 transition. The *ab initio* line-shape calculations allowed us to mitigate the expected collisional systematics reaching the 410 kHz accuracy of ν_0 . We report theoretical determination of ν_0 taking into account relativistic and QED corrections up to α^5 . Our

estimation of the accuracy of the theoretical ν_0 is 1.3 MHz. We observe 3.4σ discrepancy between experimental and theoretical ν_0 which, most likely, is caused by underestimation of the combined relativistic nonadiabatic effects. The presented method of employing the *ab initio* line-shape modeling in determination of the energies of weak molecular transitions has a large potential for further improvements (considerable improvement can be reached by using more accurate PES or by using optical cavity with higher finesse), hence it potentially could be used for addressing the problems of the proton radius [70–72] and its mass [73] or for searching for new physics [1] with the help of very accurate calculations of the molecular levels [67].

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Appendix A. Quadratic approximation of the speed dependence of γ and δ

The definition of γ_{SD} and δ_{SD} introduced in Sec. 3.3, namely, in Eqs. (25–29), is very convenient for practical purposes because, in the case of the widely used quadratic approximation,

$$\gamma(v) \approx \gamma_0 + \gamma_2 \left(\frac{v^2}{v_m^2} - \frac{3}{2} \right), \quad (\text{A.1})$$

$$\delta(v) \approx \delta_0 + \delta_2 \left(\frac{v^2}{v_m^2} - \frac{3}{2} \right), \quad (\text{A.2})$$

the γ_{SD} and δ_{SD} parameters are simply equal to γ_2 and δ_2 , respectively. Then, the physical meaning of this approximation is that the slopes of the quadratic and *ab initio* speed dependences are the same at the most probable speed v_m .

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