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Homogenization of fluid-porous interface coupling in a biconnected fractured media

Isabelle Gruais · Dan Poliševski

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Abstract The modelization of mass transfer through biconnected fractured porous media is studied by homogenizing the coupling between the Darcyan percolation and the viscous Stokes flow on their interface governed by the Beavers-Joseph law. The case of high transmission coefficients is considered. The asymptotic behaviour is completely described with the help of the solutions of some specific local problems and of a non homogeneous Neumann problem defined by the effective permeability tensor.

Keywords fractured porous media · Stokes flow · Beavers-Joseph interface · homogenization · two-scale convergence

Mathematics Subject Classification (2010) 35B27 · 76M50 · 76S05

1 Introduction

Damaged porosity is the realistic counterpart of the model of porous media that was successfully recovered by means of homogenization methods based on formal asymptotic expansions in [19] and later proved in [20] in its main part, namely the pressure problem. Variants including the influence of cracks have been studied in [3], [8] based on an adequate choice of the physical model, see [3]-[5], [11]-[14], [18] where this issue is taken into account. An important assumption in homogenization theory dealing with porous media is the periodicity of the distribution of inclusions [6], [19], [20], which has given rise to the periodicity-based two-scale convergence method of the pioneering article [16] and numerous improvements by [1], [2], [15].

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Darcy's law results from the homogenization theory applied to a periodic distribution of solid blocks immersed in a Stokes fluid. The asymptotic analysis was performed with the help of a restriction operator, which enables a natural extension of the pressure in the blocks. The next step was to consider transmission problems which proved to be a difficult question to handle in asymptotic analysis. In this direction, we have already studied (see [10]) the Stokes problem immersed in a porous medium when the interface is governed by the Beavers-Joseph law. The two-component model is that of a porous medium perforated by cracks filled with Stokes fluid interacting with the ambiant porous medium through Beavers-Joseph conditions, in the case of bounded transfer coefficient. The study is handled by the two-scale convergence method and gives rise to a coupled model between a microscopic Stokes flow and a macroscopic Darcy law.

We consider here an incompressible viscous fluid flow in a periodically structured domain consisting of two interowen regions, separated by an interface. The first region represents the system of fissures which form the connected fracture, where the viscous flow is governed by the Stokes system. The second region, which may be also connected, stands for a porous structure of a certain permeability, where the flow is governed by Darcy's law. These two flows are coupled on the interface by the Saffman's variant [18] of the Beavers-Joseph condition [5], [12] which was confirmed by [11] as the limit of a homogenization process. Besides the continuity of the normal component of the velocity, it imposes the proportionality of the tangential component of the viscous stress. The local problem has vanishing tangential velocity at the interface, which is linking both components of the corresponding microscopic cell problem while being coupled to a macroscopic Darcy law as well. The asymptotic behaviour is completely described with the help of the solutions of the local problems and of a non homogeneous Neumann problem defined by an effective permeability tensor.

The paper is organized as follows. The mathematical problem is presented in Section 2 along with a priori estimates derived from it. The coupling between the porous medium submitted to Darcy's law and the Stokes flow is described in details in Section 3. In particular, the scalings of both components are linked through the modelization of the interface which plays an important part in the asymptotic study. Compactness properties of the sequence of problems are studied in Section 4. The limit problem is introduced in Theorem 7 of Section 5 and described thereafter in Theorem 8. It consists of a macroscopic Darcy system characterized by a permeability tensor having the advantage of being easily computable through a local cell problem.

2 The properties of the two-connected components structure

Let Ω be an open connected bounded set in $\mathbb{R}^N (N \geq 2)$, locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_f be a Lipschitz open connected subset of the unit cube $Y =]0,1[^N$, such that the intersections of ∂Y_f with ∂Y are reproduced identically on the opposite faces of the cube without reaching the edges.

Repeating Y by periodicity, we assume that the reunion of all the $\bar{Y}_f$ parts, denoted by $\mathbb{R}^N_f$, is a connected domain in $\mathbb{R}^N$ with a boundary of class $\mathcal{C}^2$, such that ∂Y_f is Lipschitz-continuous. Defining $Y_s = Y \setminus \overline{Y}_f$, we assume also that the reunion of all the $\bar{Y}_s$ parts is a connected domain in $\mathbb{R}^N$. We set the origin of the coordinates system in such a point that there exists $\delta > 0$ for which $B(0,\delta) \subset \mathbb{R}^N_s = \mathbb{R}^N \setminus \overline{\mathbb{R}}^N_f$. The normal on $\Gamma = \overline{Y}_f \cap \overline{Y}_s$, outward with respect to Y_f, is denoted by ν.

For any $\varepsilon \in]0,1[$ we denote

$$Z_\varepsilon = \{k \in \mathbb{Z}^N,\ \varepsilon k + \varepsilon Y \subseteq \Omega\} \tag{2.1}$$

$$I_\varepsilon = \{k \in Z_\varepsilon, \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \forall i \in \overline{1,N}\} \tag{2.2}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

Finally, we define the system of fissures by

$$\Omega_{\varepsilon f} = \text{int}\ \left(\cup_{k\in I_\varepsilon}(\varepsilon k + \varepsilon \bar{Y}_f)\right) \tag{2.3}$$

and the porous matrix of our structure by $\Omega_{\varepsilon s} = \Omega \setminus \bar{\Omega}_{\varepsilon f}$. The interface between the porous blocks and the fluid is denoted by $\Gamma_\varepsilon = \partial\Omega_{\varepsilon f}$.

Obviously, $\Omega_{\varepsilon s}$ and $\Omega_{\varepsilon f}$ are connected and the fracture ratios of this structure have the property: $|\Omega_{\varepsilon f}|/|\Omega| \to m := |Y_f| \in]0,1[$, when $\varepsilon \to 0$.

A useful property of the present structure is the existence of a bounded extension operator (see [17]) similar to that introduced in [6] (see also [7]) in the case of isolated fractures.

Theorem 1 *There exists an extension operator $P_\varepsilon :\ H^1(\Omega_{\varepsilon f}) \to H^1(\Omega)$ such that*

$$P_\varepsilon u = u \quad in \quad \Omega_{\varepsilon f} \tag{2.4}$$

$$|e(P_\varepsilon u)|_{L^2(\Omega)} \le C|e(u)|_{L^2(\Omega_{\varepsilon f})}, \quad \forall u \in H^1(\Omega_{\varepsilon f}) \tag{2.5}$$

where C is independent of ε.

For any $h \in \{s, f\}$, we recall the main inequalities that hold here.

Lemma 1 *There exists $C > 0$ independent of ε such that*

$$|u|_{L^2(\Omega_{\varepsilon f})} \le C\varepsilon|\nabla u|_{L^2(\Omega_{\varepsilon f})}, \quad \forall u \in H^1_0(\Omega_{\varepsilon f}) \tag{2.6}$$

$$|u|_{L^2(\Omega_{\varepsilon s})} \le C(\,\varepsilon|\nabla u|_{L^2(\Omega_{\varepsilon s})} + \varepsilon^{1/2}|u|_{L^2(\partial\Omega_{\varepsilon s})}\,), \quad \forall u \in H^1(\Omega_{\varepsilon s}) \tag{2.7}$$

Theorem 2 *There exists $C > 0$ independent of ε such that*

$$|p|_{L^2(\Omega_{\varepsilon h})/\mathbb{R}} \le \frac{C}{\varepsilon}|\nabla p|_{H^{-1}(\Omega_{\varepsilon h})}, \quad \forall p \in L^2(\Omega_{\varepsilon h}). \tag{2.8}$$

Corollary 1 *There exists $C > 0$ independent of ε such that*

$$|p|_{L^2(\Omega_{\varepsilon h})/\mathbb{R}} \leq C|\nabla p|_{L^2(\Omega_{\varepsilon h})}, \quad \forall p \in H^1(\Omega_{\varepsilon h}). \tag{2.9}$$

Denoting $\chi_{\varepsilon h}(x) = \chi_h\left(\frac{x}{\varepsilon}\right)$, where χ_h is the characteristic function of Y_h in Y, $h \in \{s, f\}$, we have a specific compactness result (see [17]) for the pressure type estimates in $H^{-1}(\Omega_{\varepsilon f})$.

Theorem 3 *For every $\varepsilon \in]0,1[$, let $q^\varepsilon \in L^2(\Omega_{\varepsilon f})$, such that for some constant $C > 0$ there hold:*

$$\left|\int_{\Omega_{\varepsilon f}} q^\varepsilon(x)\, dx\right| \leq C, \quad \forall \varepsilon \in]0,1[. \tag{2.10}$$

$$|\nabla q^\varepsilon|_{H^{-1}(\Omega_{\varepsilon f})} \leq C\varepsilon, \quad \forall \varepsilon \in]0,1[. \tag{2.11}$$

Then, there exists $q \in L^2(\Omega)$ such that, on some subsequence:

$$\chi_{\varepsilon f} q^\varepsilon \overset{2}{\rightharpoonup} \chi_f q. \tag{2.12}$$

3 The coupled Darcy-Stokes percolation

A model of fluid flow through a fractured porous medium is associated to our structure by assuming that there is a filtration flow in $\Omega_{\varepsilon s}$ obeying the Darcy's law and that there is a viscous flow in $\Omega_{\varepsilon f}$ governed by the Stokes system. These two flows are coupled by a Saffman's variant [18] of the Beavers-Joseph condition [5], [12]. The system is completed by an impermeability condition on $\partial\Omega$:

$$\mathrm{div} u^\varepsilon = 0 \quad \text{in} \quad \Omega \tag{3.1}$$

$$A^\varepsilon u^\varepsilon = g - \nabla p^{\varepsilon s} \quad \text{in} \quad \Omega_{\varepsilon s}, \quad g \in L^2(\Omega) \tag{3.2}$$

$$\sigma_{ij}^{\varepsilon f} = -p^{\varepsilon f}\delta_{ij} + \varepsilon^2 e_{ij}(u^\varepsilon) \quad \text{in} \quad \Omega_{\varepsilon f} \tag{3.3}$$

$$-\frac{\partial}{\partial x_j}\sigma_{ij}^{\varepsilon f} = g_i \quad \text{in} \quad \Omega_{\varepsilon f} \tag{3.4}$$

$$v^{\varepsilon s} \cdot \nu^\varepsilon = v^{\varepsilon f} \cdot \nu^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon, \tag{3.5}$$

$$p^{\varepsilon s}\nu_i^\varepsilon + \sigma_{ij}^{\varepsilon f}\nu_j^\varepsilon + \varepsilon^r \alpha_\varepsilon(u_i^{\varepsilon f} - (u^{\varepsilon f} \cdot \nu^\varepsilon)\nu_i^\varepsilon) = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad r < 1 \tag{3.6}$$

$$u^\varepsilon \cdot n = 0 \quad \text{on} \quad \partial\Omega, \quad n \text{ the outward normal on } \partial\Omega, \tag{3.7}$$

where $u^{\varepsilon s}$, $u^{\varepsilon f}$ and $p^{\varepsilon s}$, $p^{\varepsilon f}$ stand for the corresponding velocities and pressures,

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \quad A \in (\mathcal{C}^\infty_{\mathrm{per}}(Y))^{N\times N} \quad \text{coercive, symmetric} \tag{3.8}$$

$$\alpha^\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right) \geq \alpha_0 > 0, \quad \alpha \in \mathcal{C}^1_{\mathrm{per}}(\overline{Y}) \tag{3.9}$$

and $e(v)$ denotes the symmetric tensor of the velocity gradient defined by

$$e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

As usual, we use the notations:

$$H_0(\mathrm{div}, \Omega) = \{v \in H(\mathrm{div}, \Omega), \quad v \cdot n = 0 \quad \text{on} \quad \partial\Omega\} \tag{3.10}$$

$$L_0^2(\Omega) = \{p \in L^2(\Omega), \quad \int_\Omega p = 0\} \tag{3.11}$$

$$V_0(\mathrm{div}, \Omega) = \{v \in H_0(\mathrm{div}, \Omega), \quad \mathrm{div} v = 0 \quad \text{in} \quad \Omega\} \tag{3.12}$$

Next, we define

$$H_\varepsilon = \{v \in H_0(\mathrm{div}, \Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\}, \tag{3.13}$$

the Hilbert space endowed with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon s}} u \cdot v + \int_{\Omega_{\varepsilon s}} \mathrm{div} u \, \mathrm{div} v + \varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u)\, e(v) + \varepsilon^r \int_{\Gamma_\varepsilon} (\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u)\nu^\varepsilon)\gamma^\varepsilon v, \tag{3.14}$$

where γ^ε and γ_ν^ε denote respectively the trace and the normal trace operators on Γ_ε, while ν^ε is the normal on Γ_ε, outward with respect to $\Omega_{\varepsilon f}$. The corresponding subspace of incompressible velocities is

$$V_\varepsilon = \{v \in V_0(\mathrm{div}, \Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\}. \tag{3.15}$$

Via the corresponding Korn inequality, we obtain a specific inequality:

Lemma 2 *There exists some constant $C > 0$, independent of ε, such that*

$$|u|_{L^2(\Omega_{\varepsilon f})} + \varepsilon |\nabla u|_{L^2(\Omega_{\varepsilon f})} \leq C |u|_{H_\varepsilon}, \quad \forall u \in H_\varepsilon. \tag{3.16}$$

Looking for the variational formulation of the problem (3.1)–(3.7), we define for any $u, v \in H_\varepsilon$ and $q \in L_0^2(\Omega)$

$$a_\varepsilon(u, v) := \int_{\Omega_{\varepsilon s}} A^\varepsilon u \cdot v + \varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u) : e(v) + \varepsilon^r \int_{\Gamma_\varepsilon} \alpha^\varepsilon (\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u)\nu^\varepsilon) \cdot \gamma^\varepsilon v \tag{3.17}$$

$$b_\varepsilon(q, v) := -\int_\Omega q \, \mathrm{div} v. \tag{3.18}$$

We see that if the pair $(u^\varepsilon, p^\varepsilon)$ is a smooth solution of the problem (3.1)–(3.7), then it is also a solution of the following problem:

To find $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L_0^2(\Omega)$ such that

$$a_\varepsilon(u^\varepsilon, v) + b_\varepsilon(p^\varepsilon, v) = \int_\Omega g^\varepsilon \cdot v, \quad \forall v \in H_\varepsilon, \tag{3.19}$$

$$b_\varepsilon(q, u^\varepsilon) = 0, \quad \forall q \in L_0^2(\Omega). \tag{3.20}$$

Theorem 4 *There exists a unique pair* $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L^2_0(\Omega)$ *solution of (3.19)–(3.20).*

Proof. As $H^1_0(\Omega)$ is obviously included in H_ε, the following inf-sup condition is easily satisfied by b_ε:

$$\exists C_1^\varepsilon > 0 \quad \text{such that} \quad \inf_{q \in L^2_0(\Omega)} \sup_{v \in H_\varepsilon} \frac{b_\varepsilon(q,v)}{|v|_{H_\varepsilon}|q|_{L^2_0(\Omega)}} \geq C_1^\varepsilon.$$

The positivity conditions (3.8) and (3.9) imply that

$$\exists C_2^\varepsilon > 0 \quad \text{such that} \quad a_\varepsilon(v,v) \geq C_2^\varepsilon |v|^2_{H_\varepsilon}, \quad \forall v \in H_\varepsilon,$$

that is the V_ε-ellipticity of a_ε. As we also have

$$V_\varepsilon = \{v \in H_\varepsilon, \quad b_\varepsilon(q,v) = 0, \quad \forall q \in L^2_0(\Omega)\}, \tag{3.21}$$

the proof is completed by Corollary 4.1, Ch. 1 of [9]. ■

4 A priori estimates and two-scale convergences

From now on, for any function φ defined on $\Omega \times Y$ we shall use the notations

$$\varphi^h = \varphi|_{\Omega \times Y_h}, \quad \tilde{\varphi}^h = \frac{1}{|Y_h|}\int_{Y_h} \varphi(\cdot, y)dy, \quad h \in \{s, f\}, \tag{4.1}$$

$$\tilde{\varphi} = \int_Y \varphi(\cdot, y)dy, \quad \text{that is} \quad \tilde{\varphi} = (1-m)\tilde{\varphi}^s + m\tilde{\varphi}^f. \tag{4.2}$$

Also, for any sequence $(\varphi^\varepsilon)_\varepsilon$, bounded in $L^2(\Omega \times Y)$, we denote

$$\varphi^\varepsilon \overset{2}{\rightharpoonup} \varphi$$

iff φ^ε is two-scale convergent to $\varphi \in L^2(\Omega \times Y)$ in the sense of [2].

As usual, the asymptotic study starts by the search of a priori estimates of the velocity. Noticing that $u^\varepsilon \in V_\varepsilon$ and setting $v = u^\varepsilon$ in (3.19) we get

$$|u^\varepsilon|^2_{H_\varepsilon} \leq C|u^\varepsilon|_{L^2(\Omega)} \tag{4.3}$$

and therefore:

$$|u^\varepsilon|_{L^2(\Omega_{\varepsilon s})} \leq C, \quad \varepsilon|e(u^\varepsilon)|_{L^2(\Omega_{\varepsilon f})} \leq C, \quad \varepsilon^{r/2}|\gamma^\varepsilon_\tau u^\varepsilon|_{L^2(\Gamma_\varepsilon)} \leq C \tag{4.4}$$

where, for any $v \in H^1(\Omega_{\varepsilon f})$, we define $\gamma^\varepsilon_\tau v \in H^{1/2}(\Gamma_\varepsilon)$ by

$$\gamma^\varepsilon_\tau v := \gamma^\varepsilon v - (\gamma^\varepsilon_\nu v)\nu^\varepsilon. \tag{4.5}$$

We find that $(\chi_{\varepsilon s} u^\varepsilon)_\varepsilon$, $(\chi_{\varepsilon f} u^\varepsilon)_\varepsilon$ and $\left(\varepsilon \chi_{\varepsilon f} \dfrac{\partial u^\varepsilon}{\partial x_i}\right)_\varepsilon$ are bounded in $(L^2(\Omega))^N$, $\forall i \in \{1, 2, \cdots, N\}$. Using the compacity results of [16], $\exists u \in L^2(\Omega \times Y)^N$ such that, on some subsequence

$$u^\varepsilon \overset{2}{\rightharpoonup} u \tag{4.6}$$

$$\varepsilon\chi_{\varepsilon f}\nabla u_i^\varepsilon \overset{2}{\rightharpoonup} \nabla_y u_i \tag{4.7}$$

$$u^\varepsilon \rightharpoonup \tilde{u} \in V_0(\mathrm{div}, \Omega) \quad \text{weakly in} \quad L^2(\Omega)^N. \tag{4.8}$$

Moreover, denoting

$$H_{\mathrm{per}}(\mathrm{div}, Y) = \{\varphi \in H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^N), \quad \varphi \quad \text{is } Y\text{-periodic}\} \tag{4.9}$$

$$H^1_{\mathrm{per}}(Y_f) = \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^N_f), \quad \varphi \quad \text{is } Y\text{-periodic}\} \tag{4.10}$$

we can sum the convergence results obtained until now by

Theorem 5 *There exists $u \in L^2(\Omega, H_{\mathrm{per}}(\mathrm{div}, Y))$ with the properties:*

$$u^f \in L^2(\Omega, H^1_{\mathrm{per}}(Y_f)^N), \quad \tilde{u} \in H_0(\mathrm{div}, \Omega), \tag{4.11}$$

$$\mathrm{div}_y u = 0 \quad in \quad \Omega \times Y, \tag{4.12}$$

$$\mathrm{div}\tilde{u} = 0 \quad in \quad \Omega, \tag{4.13}$$

such that the convergences (4.6) and (4.7) hold on some subsequence.

Next, we look for a priori estimates of the pressure. For this, set $v \in H_\varepsilon$ in (3.19)–(3.20) such that

$$v \in H^1_0(\Omega)^N \quad \text{and} \quad v = 0 \quad \text{in} \quad \Omega_{\varepsilon f}. \tag{4.14}$$

Using the classical L^2 decomposition, we find that $\exists p^{\varepsilon s} \in H^1(\Omega_{\varepsilon s})$ such that

$$\nabla p^{\varepsilon s} = A^\varepsilon u^\varepsilon - g^\varepsilon \in L^2(\Omega_{\varepsilon s}) \tag{4.15}$$

and recalling (4.4), we find that there exists $C > 0$, independent of ε, such that

$$|\nabla p^{\varepsilon s}|_{L^2(\Omega_{\varepsilon s})} \leq C. \tag{4.16}$$

Consequently, using Corollary 1 we have also

$$|p^{\varepsilon s} - \tilde{p}^{\varepsilon s}|_{L^2(\Omega_{\varepsilon s})} \leq C, \quad C > 0 \text{ independent of } \varepsilon. \tag{4.17}$$

It remains to estimate $\tilde{p}^{\varepsilon s}$, and implicitly $\tilde{p}^{\varepsilon f}$, as $\tilde{p}^\varepsilon = 0$. We remark here that for any $v \in H_\varepsilon$ we get

$$\frac{|\Omega|}{|\Omega_{\varepsilon f}|}\tilde{p}^{\varepsilon s}\int_{\Gamma_\varepsilon}\gamma^\varepsilon_\nu v = \int_{\Omega_{\varepsilon s}}(p^{\varepsilon s} - \tilde{p}^{\varepsilon s})\,\mathrm{div}v + \int_{\Omega_{\varepsilon f}}(p^{\varepsilon f} - \tilde{p}^{\varepsilon f})\,\mathrm{div}v - \int_\Omega gv +$$

$$+ \int_{\Omega_{\varepsilon s}} A^\varepsilon u^\varepsilon v + \varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u^\varepsilon) : e(v) + \varepsilon^r \int_{\Gamma_\varepsilon} \alpha^\varepsilon\, u^\varepsilon(v - (\gamma^\varepsilon_\nu v)\nu^\varepsilon). \tag{4.18}$$

In oder to conclude, we need a proper test function. For this, let $\theta \in H^1_0(Y)^N$ such that $\theta = \nu$ on Γ. Next, we define $\theta^\varepsilon \in H^1_0(\Omega)^N$ by

$$\theta^\varepsilon(x) = \begin{cases} \theta\left(\dfrac{x}{\varepsilon}\right) & \text{for } x \in \Omega_\varepsilon = \mathrm{int}\ \left(\cup_{k\in I_\varepsilon}(\varepsilon k + \varepsilon\bar{Y})\right) \\ 0 & \text{for } x \in \Omega \setminus \bar{\Omega}_\varepsilon; \end{cases} \tag{4.19}$$

Setting $v = \varepsilon\theta^\varepsilon$ in (4.18), we finally get

$$|\tilde{p}^{\varepsilon s}| \leq C|\Gamma_\varepsilon|^{-1}\left(1 + |\theta^\varepsilon|_{H^1(\Omega)}\right), \tag{4.20}$$

that is,

$$(\tilde{p}^{\varepsilon s})_\varepsilon \quad \text{and} \quad (\tilde{p}^{\varepsilon f})_\varepsilon \quad \text{are bounded.} \tag{4.21}$$

Consequently, there exists $C > 0$ independent of ε, such that

$$|p^{\varepsilon s}|_{H^1(\Omega_{\varepsilon s})} \leq C \tag{4.22}$$

and using the compactness results of [2], there exists $p^s \in H^1(\Omega)$ such that

$$\chi_{\varepsilon s} p^\varepsilon \overset{2}{\rightharpoonup} \chi_s p^s, \quad \text{on some subsequence.} \tag{4.23}$$

Next, we set $\psi \in H_\varepsilon$ in (3.19)–(3.20) such that $\psi = 0$ in $\Omega_{\varepsilon s}$. It follows that $\exists p^{\varepsilon f} \in L^2(\Omega_{\varepsilon f})$ such that

$$\langle \nabla p^{\varepsilon f}, \psi\rangle_{H^{-1}(\Omega_{\varepsilon f})} = \varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u^\varepsilon)\, e(\psi) - \int_{\Omega_{\varepsilon f}} g^\varepsilon \psi, \quad \forall \psi \in H_0^1(\Omega_{\varepsilon f})^N. \tag{4.24}$$

Consequently, recalling (4.4) and (4.21) we find that $\exists C > 0$, independent of ε, such that

$$|\nabla p^{\varepsilon f}|_{H^{-1}(\Omega_{\varepsilon f})} \leq C\varepsilon. \tag{4.25}$$

In the light of (4.21), (4.25) and applying Theorem 3 we find that there exists $p^f \in L^2(\Omega)$ such that

$$\chi_{\varepsilon f} p^\varepsilon \overset{2}{\rightharpoonup} \chi_f p^f, \quad \text{on some subsequence.} \tag{4.26}$$

Lemma 3 *The limits in (4.23) and (4.26) satisfy:*

$$p^f = p^s \quad \textit{in} \quad L^2(\Omega). \tag{4.27}$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{C}^\infty_{\text{per}}(Y)^N$ such that

$$\psi = (\gamma_\nu \psi)\nu \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Gamma \gamma_\nu \psi \neq 0.$$

Set

$$v_i^\varepsilon(x) = \varepsilon\varphi(x)\psi_i\left(\frac{x}{\varepsilon}\right), \quad \forall i \in \{1, \cdots, N\},$$

in the variational formulation (3.19)–(3.20). Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_\Gamma \gamma_\nu \psi \, d\sigma \int_\Omega (p^f(x) - p^s(x))\varphi(x)dx = 0, \ \forall \varphi \in \mathcal{D}(\Omega),$$

which concludes the proof. ■

The convergence results concerning the pressure can be summed up by

Theorem 6 *There exists $p \in \tilde{H}^1(\Omega) := L^2_0(\Omega) \cap H^1(\Omega)$ such that on some subsequence we have*

$$p^\varepsilon \overset{2}{\rightharpoonup} p. \tag{4.28}$$

A characteristic property issued from the Beavers-Joseph transfer condition is given by

Lemma 4 *The limit u satisfies:*

$$\gamma u^f = (\gamma_\nu u^f)\nu \quad on \quad \Omega \times \Gamma. \tag{4.29}$$

Proof. For $\varphi \in \mathcal{D}(\Omega, H^1_{\text{per}}(Y_f)^N)$ we define

$$I_\varepsilon(\varphi) = \varepsilon \int_{\Gamma_\varepsilon} \alpha^\varepsilon \gamma^\varepsilon u^\varepsilon (\gamma^\varepsilon \varphi - (\gamma^\varepsilon_\nu \varphi)\nu^\varepsilon). \tag{4.30}$$

As α and Γ are assumed smooth enough, there exists $\Phi \in \mathcal{C}^\infty(\Omega, C^1_{\text{per}}(Y_f))^{N\times N}$ such that

$$\Phi_{ij}(x,y) = \alpha(y)(\varphi_i(x,y) - (\varphi(x,y)\cdot\nu(y))\nu_i(y))\nu_j(y) \quad \text{on} \quad \Omega \times \Gamma. \tag{4.31}$$

Using the notation

$$\Phi^\varepsilon(x) = \Phi\left(x, \frac{x}{\varepsilon}\right), \tag{4.32}$$

we find that

$$I_\varepsilon(\varphi) = \varepsilon \int_{\partial\Omega_{\varepsilon f}} \Phi^\varepsilon_{ij} u^\varepsilon_i \nu^\varepsilon_j d\sigma =$$

$$= \int_{\Omega_{\varepsilon f}} \left(\frac{\partial \Phi^\varepsilon_{ij}}{\partial y_j}\right)^\varepsilon u^\varepsilon_i + \varepsilon \int_{\Omega_{\varepsilon f}} \left(\frac{\partial \Phi_{ij}}{\partial x_j}\right)^\varepsilon u^\varepsilon_i + \varepsilon \int_{\Omega_{\varepsilon f}} \Phi^\varepsilon_{ij} \frac{\partial u^\varepsilon_i}{\partial x_j} \tag{4.33}$$

and a direct computation yields

$$\lim_{\varepsilon\to 0} I_\varepsilon(\varphi) = \int_{\Omega\times Y_f} \left(\frac{\partial \Phi_{ij}}{\partial y_j} u^f_i + \Phi_{ij}\frac{\partial u^f_i}{\partial y_j}\right) = \int_{\Omega\times Y_f} \Phi_{ij} u^f_i \nu_j =$$

$$= \int_{\Omega\times\Gamma} \alpha\gamma u^f(\gamma\varphi - (\gamma_\nu\varphi)\nu). \tag{4.34}$$

Setting $v = \varepsilon^{1-r}\varphi$ as test function in (3.19) and passing to the limit with $\varepsilon \to 0$ we find that

$$\lim_{\varepsilon\to 0} I_\varepsilon(\varphi) = 0 \tag{4.35}$$

that is,

$$\int_{\Omega\times\Gamma} \alpha\gamma u^f(\gamma\varphi - (\gamma_\nu\varphi)\nu)) = 0, \quad \forall \varphi \in \mathcal{D}(\Omega, H^1_{\text{per}}(Y_f)^N), \tag{4.36}$$

which yields the result. ■

5 The two-scale homogenized problem

Here we find the so-called two-scale homogenized problem, verified by the limits u and p, given by Theorems 5 and 6. By proving that this problem is well-posed, it turns out that all the convergences in Theorems 5, 6 and so forth hold on the entire sequence. We conclude that the asymptotic behaviour of $(u^\varepsilon, p^\varepsilon)$ is completely described by (u, p), the unique solution of the homogenized problem.

Denoting

$$\gamma_\tau v = \gamma v - (\gamma_\nu v)\nu, \quad \forall v \in H^1(Y_f), \tag{5.1}$$

we introduce the Hilbert space

$$X = \{v \in L^2(\Omega, H_{\text{per}}(\text{div}, Y),\ v^f \in L^2(\Omega, H^1_{\text{per}}(Y_f)^N),\ \tilde{v} \in H_0(\text{div}, \Omega),$$
$$\text{div}_y v = 0 \quad \text{in} \quad \Omega \times Y, \quad \gamma_\tau v^f = 0 \quad \text{on} \quad \Omega \times \Gamma\}, \tag{5.2}$$

endowed with the scalar product

$$(u, \varphi)_X = \int_{\Omega \times Y_s} u \cdot \varphi + \int_\Omega \text{div}\tilde{u}\, \text{div}\tilde{\varphi} + \int_{\Omega \times Y_f} e_y(u) : e_y(\varphi). \tag{5.3}$$

Introducing

$$X_0 = \{v \in X, \quad \text{div}\tilde{v} = 0 \quad \text{in} \quad \Omega\}, \tag{5.4}$$

$$M = \tilde{H}^1(\Omega). \tag{5.5}$$

$$b(q, v) = -\int_\Omega q\, \text{div}\tilde{v}, \quad \forall (q, v) \in M \times X, \tag{5.6}$$

we consider the problem:

To find $(v, q) \in X \times M$ such that

$$a(v, \varphi) + b(q, \varphi) = \int_\Omega g \cdot \tilde{\varphi}, \quad \forall \varphi \in X \tag{5.7}$$

$$b(\pi, v) = 0, \quad \forall \pi \in M \tag{5.8}$$

where b is given by (5.6) and a is defined by

$$a(v, \varphi) = \int_{\Omega \times Y_s} Av \cdot \varphi + \int_{\Omega \times Y_f} e_y(v) : e_y(\varphi). \tag{5.9}$$

Lemma 5 *The problem (5.7)–(5.8) is well-defined.*

Proof. First, we notice that

$$X_0 = \{v \in X, \quad b(q, v) = 0, \quad \forall q \in M\}. \tag{5.10}$$

As the X_0-ellipticity of a in X is obvious, the proof is completed by Corollary 4.1, Ch.1 of [9], the corresponding inf-sup condition being justified via the surjectivity of div $\in \mathcal{L}(H^1_0(\Omega)^N, L^2_0(\Omega))$. ■

The fact that (5.7)–(5.8) is the homogenized problem of the present case is justified below.

Theorem 7 *u and p introduced by Theorems 5 and 6 form a pair $(u,p) \in X \times M$ which is the unique solution of the problem (5.7)–(5.8).*

Proof. Let $\varphi \in X \cap \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))^N$. Denoting, as usual, $\varphi^\varepsilon(x) = \varphi\left(x, \dfrac{x}{\varepsilon}\right)$, we can set $v(x) = \varphi^\varepsilon(x)$ in (3.19). Passing to the limit with $\varepsilon \to 0$, taking advantage of the property $\gamma^\varepsilon_\tau \varphi^\varepsilon = 0$ by construction and using the two-scale convergences of Theorems 5 and 6, we obtain:

$$b_\varepsilon(p^\varepsilon, \varphi^\varepsilon) \to -\int_{\Omega \times Y_s} p\,(\text{div}_x \varphi) = b(p, \varphi),$$

and

$$a_\varepsilon(u^\varepsilon, \varphi^\varepsilon) \to a(u, \varphi),$$

which concludes the proof. ■

The solution of the homogenized problem (5.7)–(5.8) has a further description which takes into account the so-called local problems and effective coefficients. For this, let us introduce the unique solution of the Darcy-Stokes system in Y, with the classical normal flow coupling conditions:

For every $i \in \{1, \cdots, N\}$, $w^i \in W$ is the solution of the problem:

$$\int_{Y_s} A w^i \psi + \int_{Y_f} e_y(w^i)\, e_y(\psi) = \int_Y \psi_i, \quad \forall \psi \in W \tag{5.11}$$

where

$$W = \{w \in H_{\text{per}}(\text{div}, Y),\ w^f \in H^1_{\text{per}}(Y_f)^N,\ \text{div} w = 0 \text{ in } Y,\ \gamma_\tau w^f = 0 \text{ on } \Gamma\}. \tag{5.12}$$

Next, let $q \in \tilde{H}^1(\Omega)$ be the unique solution of the following compatible non-homogeneous Neumann problem:

$$\text{div}(K \nabla q) = \text{div}(Kg) \quad \text{in} \quad \Omega, \tag{5.13}$$

$$K \nabla q \cdot n = Kg \cdot n \quad \text{on} \quad \partial\Omega, \tag{5.14}$$

where K is the symmetric and positive tensor defined by

$$K_{ji} = \int_Y w^i_j. \tag{5.15}$$

Taking into account that the homogenized problem (5.7)–(5.8) is well-posed, then by straightforward verification we obtain the complete asymptotic behaviour of the solution:

Theorem 8 *If $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L^2_0(\Omega)$ is the solution of (3.19)–(3.20), then*

$$u^\varepsilon \overset{2}{\rightharpoonup} u = \left(g_i - \frac{\partial q}{\partial x_i}\right) w^i, \tag{5.16}$$

$$p^\varepsilon \overset{2}{\rightharpoonup} p = q, \tag{5.17}$$

where $w^i \in W$ and $q \in \tilde{H}^1(\Omega)$ are defined by (5.11) and (5.13)–(5.14).

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