Sand dunes as migrating strings

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We develop a reduced complexity model for three-dimensional sand dunes, based on a simplified description of the longitudinal and lateral sand transport. The spatiotemporal evolution of a dune migrating over a nonerodible bed under unidirectional wind is reduced to the dynamics of its crest line, providing a simple framework for the investigation of three-dimensional dunes, such as barchan and transverse dunes. Within this model, we derive analytical solutions for barchan dunes and investigate the stability of a rectilinear transverse dune against lateral fluctuations. We show, in particular, that the latter is unstable only if the lateral transport on the dune slip face prevails over that on the upwind face. We also predict the wavelength and the characteristic time that control the subsequent evolution of an unstable transverse dune into a wavy ridge and the ultimate fragmentation into barchan dunes.

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I. INTRODUCTION

Aeolian sand dunes are ubiquitous natural patterns present on Earth but also on other planets, such as Mars. The mechanism responsible for the primary instability that leads a flat sand bed to destabilize in favor of a sand wave under unidirectional flow is now well identified [1,2]. The instability results from the phase lag between the flow shear stress and the bedform, whereas, the most unstable wavelength, which sets the minimum size for a dune, is governed by the drag length (i.e., the length needed for the sediment to equilibrate with the flow) [1,2]. The secondary instabilities, such as those leading to three-dimensional (3D) patterns remain, however, poorly understood.

Here, we address the issue on the stability of a rectilinear transverse dune migrating over a nonerodible bed under unidirectional flow and its subsequent dynamics. Recently, rescaled water tank experiments [3] show that a rectilinear transverse dune is intrinsically unstable and evolves into a wavy ridge that eventually breaks up into barchan dunes. Numerical studies [3,4] based on the resolution of Reynolds equations and coupled with a sediment transport model have shown their ability to replicate the complex dynamic of a transverse dune.

In this article, we develop, instead, a simple model based on a reduced number of physical ingredients which allows clearly identifying the mechanisms competing for the destabilization of a rectilinear transverse dune. We derive an explicit criterion for the stability onset of rectilinear transverse dunes and provide analytical predictions concerning the long-term dynamics of the crest line and its subsequent fragmentation into barchan dunes. The developed model further allows for providing analytical solutions for barchan dunes.

In Sec. II, we set out the theoretical framework of our model. In Sec. III, we investigate the linear stability analysis of rectilinear transverse dunes, whereas, in Sec. IV, we provide analytical solutions for barchan dunes. Section V presents the model predictions concerning long-term dynamics of transverse dunes. Finally, a conclusion is given in Sec. VI.

II. MODEL

Numerical models [4–7] are generally based on the resolution of the flow over the bedform to provide the local basal shear stress. The flux in sediment is then deduced via the implementation of a transport law linking the mass transport rate to the local basal shear stress, and the bedform evolution can be computed using the mass conservation equation. Due to flow separation, the downwind slope is a region with no flow and captures the sand transported from the upwind slope. The angle of the downwind slope is limited by a threshold value above which avalanches occur and transport sediment along the steepest slope. A crucial feature of the modeling of the 3D bedform is the partition of the mass flux between the longitudinal and the lateral directions. In aeolian transport, the saltation mode (i.e., ballistic motion of the highly energetic grains through successive jumps) mainly contributes to the longitudinal transport, whereas, the reptation mode (stochastic motion of the grains ejected from the bed via the collision of the saltation grains with the bed during the splash process) participate in the lateral transport [5].

Our model is based on several simplifications in comparison with these numerical models. The first simplification is to disregard the complex flow over a bedform and to assume a simple triangular geometry for the bedform cross section (see Fig. 1). The angles of the upwind and downwind slopes (i.e., θ and ϕ, respectively) are set to typical values of aeolian dunes: $\theta = 7^\circ$ and $\phi = 26^\circ$ [8]. A dune cross section is then uniquely determined by the streamwise position x and height h of its summit. A transverse dune can be, thus, simply described in terms of only two continuum variables $x(y,t)$ and $h(y,t)$, where y represents the lateral location of the cross section and t represents the time.

The second simplification concerns the description of the sediment transport. The longitudinal transport is characterized by two features: (i) the incoming sand flux $f_{\rm in}$, that feeds the dune, and (ii) the sand flux q at the dune crest, which is primarily fixed by the flow strength. Both fluxes are prescribed by the upwind boundary conditions. The sand advected up to the crest is ultimately captured by the slip face with an

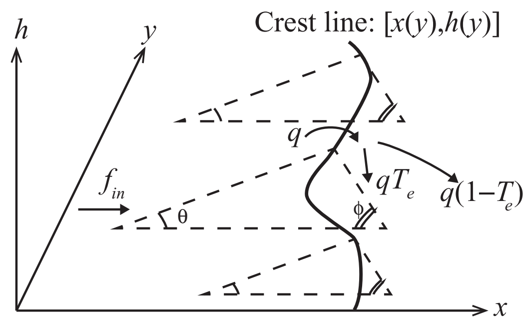

FIG. 1. Schematic of the crest line model. Upwind and downwind slope angles are θ and ϕ. Dune crest coordinates are (x,h). The incoming flux of sediment is f_{in}, the sand flux at the crest is q, and the sand trapping efficiency of the slip face is T_e.

efficiency rate T_e [9]. In most numerical models [4–7], the efficiency rate is taken to be 1 considering a fully efficient capture by the slip face. However, as shown in Ref. [9], for a given flow strength, the capture rate is an increasing function of the dune height h. At this stage, we will consider, for the sake of generality, that T_e is an arbitrary function of h.

The modeling of the sediment transport should be complemented by the lateral transport. The latter is characterized by lateral mass fluxes on the upwind and downwind slopes, denoted by J_u and J_d, respectively. Following Niiya *et al.* [10], we will assume that lateral fluxes are governed by the transverse gradients of the upwind and downwind dune faces. Integrated over each dune face, we obtain the following expressions for J_u and J_d:

$$J_u = D_u\left(-\frac{B}{A}h\frac{\partial h}{\partial y} + h\frac{\partial x}{\partial y}\right), \qquad (1)$$

$$J_d = D_d\left(-\frac{C}{A}h\frac{\partial h}{\partial y} - h\frac{\partial x}{\partial y}\right), \qquad (2)$$

where A, B, and C are geometrical constants [$A = \tan\theta\,\tan\phi/(\tan\theta + \tan\phi) = 0.1$, $B = \tan\phi/(\tan\theta + \tan\phi) = 0.8$, and $C = \tan\theta/(\tan\theta + \tan\phi) = 0.2$]. D_u and D_d are constant coefficients characterizing the intensity of the lateral diffusion on the upwind and downwind slopes, respectively.

Within this framework, the temporal evolution of the dune crest is simply obtained from mass balance and is given by

$$h\frac{\partial x}{\partial t} = q(BT_e + C) - Cf_{\text{in}} - B\frac{\partial J_d}{\partial y} + C\frac{\partial J_u}{\partial y}, \qquad (3)$$

$$h\frac{\partial h}{\partial t} = A\left[f_{\text{in}} - (1 - T_e)q - \frac{\partial J_d}{\partial y} - \frac{\partial J_u}{\partial y}\right]. \qquad (4)$$

The system of Eqs. (3) and (4) combined with Eqs. (1) and (2) completely describes the dune dynamics.

III. LINEAR STABILITY OF RECTILINEAR TRANSVERSE DUNES

The model equations have a simple time-invariant solution corresponding to a rectilinear transverse dune that drifts at a constant velocity: $x \equiv x_0 = V_0 t$ with $V_0 = qT_e(h_0)/h_0$. The height h_0 of the dune is fixed by the incoming flux: $f_{\text{in}} = [1 - T_e(h_0)]q$. Note that, in the situation where $f_{\text{in}} = 0$, there exists a stationary solution only if $T_e = 1$. In that case, the dune can be of arbitrary height.

To investigate the stability of the dune against transverse perturbations, we perform a linear stability analysis. We assume sinusoidal perturbations of the form

$$x(y,t) = x_0(t) + x_1 e^{iky+\omega t}, \qquad (5)$$

$$h(y,t) = h_0 + h_1 e^{iky+\omega t}, \qquad (6)$$

where k and ω, respectively, are the wave number and the growth rate of the perturbation. Plugging Eqs. (5) and (6) into Eqs. (3) and (4) and retaining only the linear terms in x_1 and h_1, we get the most unstable mode ω_+,

$$\begin{aligned} 2h_0^2\omega_+ = {} & Aq\,\delta T_e - D_u(1+\rho)h_0^2k^2 + \big[A^2q^2\delta^2T_e^2 \\ & + 2AqT_eD_u(\rho - 1)(2-\delta)h_0^2k^2 \\ & + D_u^2(1-\rho)^2h_0^4k^4\big]^{1/2}. \end{aligned} \qquad (7)$$

Note that we introduced two dimensionless parameters,

$$\delta = h_0\frac{T_e'(h_0)}{T_e(h_0)}, \qquad (8)$$

$$\rho = \frac{D_d}{D_u}, \qquad (9)$$

where the prime in Eq. (8) stands for the derivation with respect to h. δ quantifies the variation in the trapping efficiency with the dune height, whereas, ρ quantifies the relative magnitude of the lateral diffusion on the downwind slope with that on the upwind face.

The values of these parameters are found to be crucial for the dune stability. A positive δ acts as a destabilizing effect on the system. In particular, a homogeneous deformation (i.e., $k = 0$) is intrinsically unstable for $\delta > 0$ ($\omega_+ \propto qT_e\delta > 0$). A slight uniform increase (respectively, decrease) in the dune height will be, thus, amplified as more and more sand will be captured by the slip face (respectively, less and less). Conversely, a negative δ has a stabilizing effect. The differential lateral transport on upwind and downwind slopes is a key mechanism for the dune stability. Whereas, lateral transport on the upwind slope always plays a stabilizing role, lateral transport on the downwind slope will amplify crest line perturbations. If we imagine that a slice of the dune is in advance with respect to the rest of the dune, lateral downwind transport will cause the volume of the slice to decrease, resulting in an increase in the slice migration speed and, thus, reinforcing the perturbation.

A quantitative analysis of the dispersion relation Eq. (7) allows for determining the domain stability of the transverse dune in the parameter space (δ,ρ) (see Fig. 2). (i) For $\delta > 0$, there always exists a band of unstable modes k between 0 and a cutoff value k_c, causing a rectilinear transverse dune to be unstable for any value of ρ [see Fig 3(a)]. It reflects the destabilizing effect of an increasing with height capture rate. (ii) In contrast, for $\delta \leqslant 0$, a band of unstable modes appears only if downwind lateral diffusion prevails over upwind lateral diffusion [or, more precisely, if ρ is greater than a critical value given by $\rho_c = 1 - \delta$; see Fig. 3(b)]. Consequently, a rectilinear dune will be stable if $\rho < \rho_c$ and will be unstable otherwise. The stabilizing effect of the capture rate is overcome by the destabilizing effect of lateral transport in the slip face.

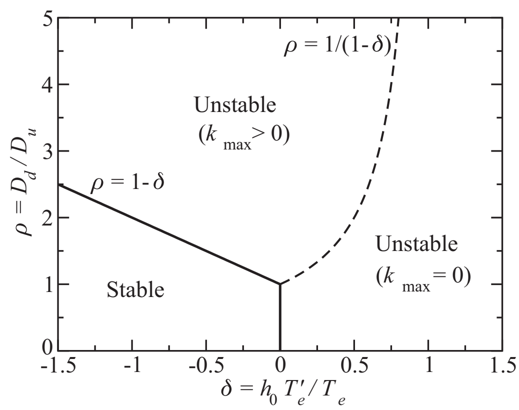

FIG. 2. Stability diagram of a rectilinear transverse dune in the parameter space (δ,ρ). In the case where $\delta > 0$, rectilinear transverse dunes are always unstable. Conversely, in the case where $\delta \leqslant 0$, rectilinear transverse dunes are stable only if $\rho \leqslant 1-\delta$. The dashed line delimits the unstable regimes where the most unstable mode k_{max} is finite from the regimes where the latter is reduced to zero.

Interestingly, the most unstable mode k_{max} either is reduced to zero [for $\delta > 0$ and $\rho < 1/(1-\delta)$] or is finite otherwise (see Fig. 2). We, thus, expect two different scenarios for the dune crest dynamics: If $k_{max} = 0$, we expect the dune height to uniformly increase or decrease at a much faster rate than the unstable modes with a finite wave number, whereas, if $k_{max} > 0$, the crest is expected to develop a wavy shape with a definite wavelength $\lambda_{max} = 2\pi/k_{max}$ within a characteristic time $\tau_{max} = \omega_+^{-1}(k_{max})$ given by

$$\lambda_{max} = 2\pi h_0 \sqrt{\frac{D_u}{AqT_e}\frac{\rho(\rho-1)}{(1+\rho)\sqrt{\rho}\sqrt{1-\delta}-\rho(2-\delta)}}, \quad (10)$$

$$\tau_{max} = \frac{h_0^2}{AqT_e}\frac{\rho-1}{(\sqrt{\rho}-\sqrt{1-\delta})^2}. \quad (11)$$

These results on the transverse dune stability are extremely general and are consistent with the predictions of previous theoretical models and numerical simulations obtained in more restricted configurations. The simulation of Parteli *et al.* [4] and the analytical model of Melo *et al.* [11], based both on a fully efficient sand capture at the slip face (i.e., with $T_e = 1$ and $\delta = 0$) and on a lateral transport dominated by the avalanches in the downwind slope (i.e., with $\rho \gg 1$), predict that a transverse dune is intrinsically unstable as expected from our stability analysis. The new finding of our study is that there exists a region in the parameter space of our model where a straight transverse dune is expected to be stable.

The next important issue is to determine the range of possible values for the model parameters T_e, D_u, and D_d in real physical situations. As shown by Momiji and Warren [9], the efficiency of the capture rate T_e for aeolian dunes is an increasing function of the dune height ($\delta > 0$) given the flow strength. Situations where the capture rate would decrease with increasing height ($\delta > 0$) are unlikely to occur for aeolian dunes. It is, however, not excluded that, in other contexts, such as for subaqueous dunes, the interaction of the bedform with the free surface of the flow may alter the sand capture process at the slip face leading to a negative δ. The capture becomes fully efficient (i.e., $T_e = 1$) for dunes with large enough slip face. Following these arguments, the simplest law for T_e can be written as

$$T_e = \begin{cases} h/h_c, & \text{if} \quad h \leqslant h_c, \\ 1, & \text{if} \quad h > h_c, \end{cases} \quad (12)$$

where h_c is the critical dune height above which $T_e = 1$. From Eq. (12), the parameter δ is easily calculated. For large dunes (i.e., larger than h_c), $\delta = 0$, whereas, for small dunes (i.e., smaller than h_c), $\delta = 1$. (Had we taken a nonlinear law for T_e, δ would have been different from 1 for small dunes but still positive and finite). From the stability diagram (see Fig. 2), we, thus, conclude that small dunes are unstable whatever the value of ρ. In contrast, large dunes are expected to be unstable only if ρ is greater than 1, or in other words, if the transverse transport is greater on the downwind slope than on the upwind slope. This is one of the important predictions of our analysis.

It appears, therefore, crucial for large dunes to determine in which situations downwind lateral transport prevails over upwind lateral transport. It is a difficult task since lateral transport is driven by many intricate phenomena (avalanches in the downwind slope, wind deflection, saltation, and gravity in the upwind slope). Most transverse sand dunes formed under unidirectional flows (either in air or in water [3,12]) exhibit sinuous or irregular crest lines, indicating that upstream lateral transport is weaker than downstream lateral transport and is unable to stabilize the crest line. However, one cannot exclude that there might exist flow configurations (not yet explored) for which straight transverse dunes are stable. A wide variety

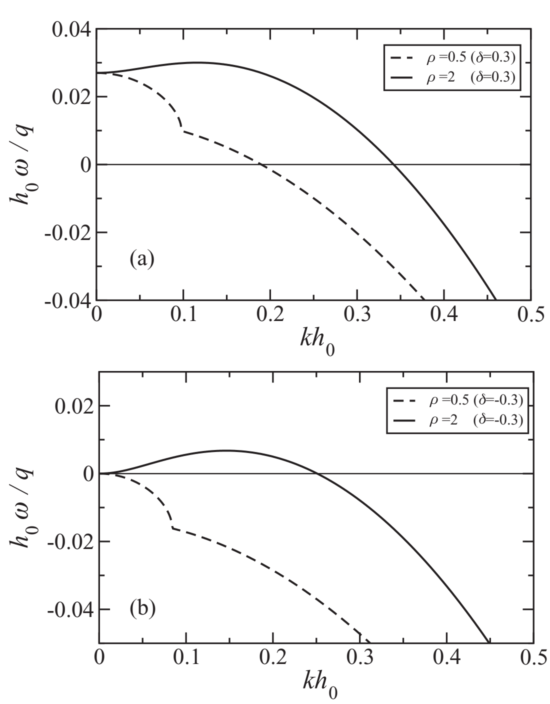

FIG. 3. Growth rate ω versus k for various values of δ and ρ: (a) $\delta = 0.3$ and $\rho = 0.5,2$; (b) $\delta = 0$ and $\rho = 0.5,2$. We set $q/D_u = 0.5$ and $T_e = 0.9$. For $\delta = 0.3$, there is a band of unstable wave numbers whatever the value ρ. For $\delta = 0$, a band of unstable modes appear for $\rho > 1 - \delta = 1$.

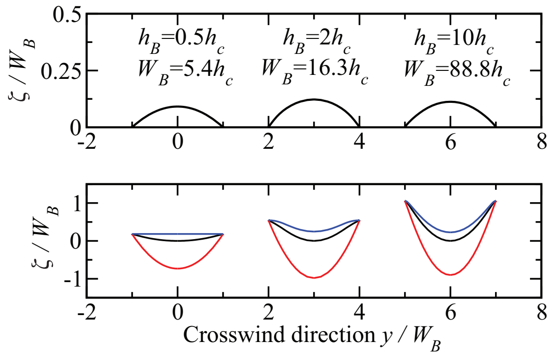

FIG. 4. (Color online) Height profile $\zeta(x)$ and crest line profile $\xi(y)$ for barchan dunes of various heights h_B rescaled by the barchan half-width W_B. Model parameters: $\rho = 2$ and $q = D_u$.

of processes could enhance the upstream lateral transport, such as flow vortices, or fluctuations in the flow strength but are unfortunately poorly documented [12]. It would be, thus, interesting to conduct, for example, rescaled water tank experiments within flow conditions that would favor upstream lateral transport to check our predictions.

IV. BARCHAN DUNES

Our model can also provide solutions corresponding to crescentic barchan dunes as a time-invariant solution drifting with a constant speed under a uniformly distributed incoming flux $f_{\rm in}$: $x(y,t) = V_B\,t + \xi(y)$ and $h(y,t) = \zeta(y)$. Assuming that the capture rate $T_e(h)$ obeys Eq. (12), the height and position profile can be obtained via a numerical resolution of the model equations for a steadily migrating state and a uniformly distributed incoming flux $f_{\rm in}$ (see Fig. 4).

In the limit case where the barchan dune height h_B is much greater than h_c, analytical predictions can be derived. We find, in particular, that the migration speed is inversely proportional to the dune height,

$$V_B = \frac{3q}{2h_B}, \tag{13}$$

and that the dunes have parabolic profiles given by

$$\xi(y) = \kappa_\xi\, y^2, \tag{14}$$

$$\zeta(y) = h_B - \kappa_\zeta\, y^2, \tag{15}$$

with

$$\kappa_\xi = \frac{q(B + C\rho)}{4D_d h_B}, \tag{16}$$

$$\kappa_\zeta = \frac{Aq(\rho - 1)}{4D_d h_B}. \tag{17}$$

Equation (17) tells us that the barchan half-width $W_B(\equiv\sqrt{h_B/\kappa_\zeta})$ scales linearly with its height h_B,

$$W_B = \sqrt{\frac{4D_d}{Aq(\rho - 1)}}\,h_B. \tag{18}$$

Several remarks follow. First, the parabolic profiles as well as the scalings found for the migration speed and the barchan width are in agreement with field observations [8,13]. Second, large barchan dunes exist only if $\rho > 1$, that is, in the parameter space where large transverse dunes are unstable against lateral fluctuations. Third, the analytical expressions of the curvatures κ_ζ and κ_ξ can be utilized to infer, from geometrical properties of real barchan dunes, an estimation of the diffusion coefficient D_d and D_u. Exploiting the topographic measurements on barchan dunes reported by Sauermann *et al.* [8], we get $D_d \approx 250$ and $D_u \approx 100$ m^2/yr such that $\rho \approx 2.5$ (see the Appendix).

Another interesting feature is the outgoing flux $f_{\rm out}$ escaping from the barchan dune. In an equilibrium regime (i.e., stationary), we simply have $\langle f_{\rm out}\rangle = f_{\rm in}$ where $\langle\,\rangle$ means the average over the lateral extent of a barchan. The latter can be derived analytically for large barchan dunes ($h_B \gtrsim h_c$),

$$\langle f_{\rm out}\rangle \approx q\sqrt{\rho/6(\rho-1)}(W_c/2W_B) \quad \text{for } W_B \gtrsim W_c, \tag{19}$$

where W_B is the half-width of the barchan dune. W_c corresponds to the half-width of the barchan dune of height h_c ($W_c = \sqrt{6D_u/Aq}\,h_c$). The outgoing flux is a decreasing function of the barchan width in agreement with the field observation [2]. As a consequence, barchan dunes are intrinsically unstable against variations in their widths.

V. LONG-TERM DYNAMICS OF TRANSVERSE DUNES

Long-term dynamics of the transverse dune can be investigated through the numerical integration of the model equations. For the numerical investigations, we assume that $T_e(h)$ is given by Eq. (12). As expected from the linear stability analysis, different scenarios occur according to the initial height h_0 of the transverse dune.

For small transverse dunes (i.e., $h_0 < h_c$), we observe, depending on the initial perturbation, either an increase or a decrease in the average dune height, which ultimately leads either to a giant transverse dune or to a complete disappearance of the latter. This scenario is in agreement with the prediction of the linear stability analysis since, for small dunes, $k_{\rm max} = 0$.

Larger transverse dunes (i.e., $h_0 > h_c$) become unstable only if $\rho > 1$. In this case, they develop a wavy crest line with a finite wavelength corresponding to the linearly most unstable mode $k_{\rm max}$. The line crest undulation is accompanied by an antiphase modulation in the dune height (in the crosswind direction) whose amplitude increases exponentially in the course of time as predicted by the linear analysis. Then comes a time when the lower parts of the crest line approach the ground ultimately leading the transverse dune to fragment into a multitude of individual barchan dunes (see Fig. 5). The initial width of the emerging barchan dunes is governed by the most unstable wavelength $\lambda_{\rm max}$ ($\lambda_{\rm max} \approx 18h_0\sqrt{D_u/Aq}$ for $\rho = 2$ and $\delta = 0$), and the characteristic time for barchans to emerge scales as $\tau_{\rm max}$, the inverse of the growth rate of the most dangerous mode ($\tau_{\rm max} \approx 6h_0^2/Aq$ for $\rho = 2$ and $\delta = 0$). The further evolution is characterized by the decrease in the barchan width in the course of time until complete disappearance because barchan solutions are intrinsically unstable. This scenario corresponds to that described experimentally by Reffet *et al.* [3] and was confirmed later by Parteli *et al.* [4] through numerical simulations.

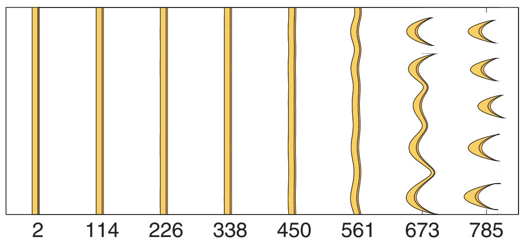

FIG. 5. (Color online) Temporal evolution of a rectilinear transverse dune with an initial height $h_0 > h_c$, ultimately leading to barchan dunes. The lateral extent of the transverse dune is $L = 5\lambda_{\max}$, and the crest line position is initially perturbed with a random signal whose amplitude x_1 is such that $|x_1| < 0.01h_0$. Model parameters: $\rho = 2$, $q = D_u = 1$, and $h_0 = 5h_c$.

VI. CONCLUSION

The crest line model is a simple alternative model to investigate the dynamics of sand dunes migrating on a bedrock under unidirectional flow. It provides a unique framework to derive analytical solutions for barchan dunes and a stability criterion for rectilinear transverse dunes. In particular, we predict the wavelength driving the destabilization process of a rectilinear transverse dune into barchan dunes. This prediction could be tested, for example, through rescaled laboratory experiments. Further interesting issues could be tackled within this model, such as the long-term dynamics of a transverse dune under time-dependent boundary conditions or the spatiotemporal evolution of a train of transverse dunes. In the future, it would also be worthwhile to extend the model to more complex wind regimes with various wind directions.

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APPENDIX: CALCULATION OF THE LATERAL DIFFUSION COEFFICIENTS FROM GEOMORPHOLOGICAL DATA

Here, we detail how we can infer the values of the lateral diffusion coefficients D_u and D_d introduced in our model from the geometrical parameters of a barchan dune. As illustrated in Fig. 6, several morphological parameters can be extracted from field measurements, such as the dune height h_B, the half-width W_B, the length of the upwind face L_0, the length of the slip face L_s, and the length of the barchan horns L_c.

We recall that our model predicts that large barchan dunes (i.e., $h_B \gg h_c$) have parabolic profiles given by

$$\xi(y) = \kappa_\xi y^2, \tag{A1}$$

$$\zeta(y) = h_B - \kappa_\zeta y^2, \tag{A2}$$

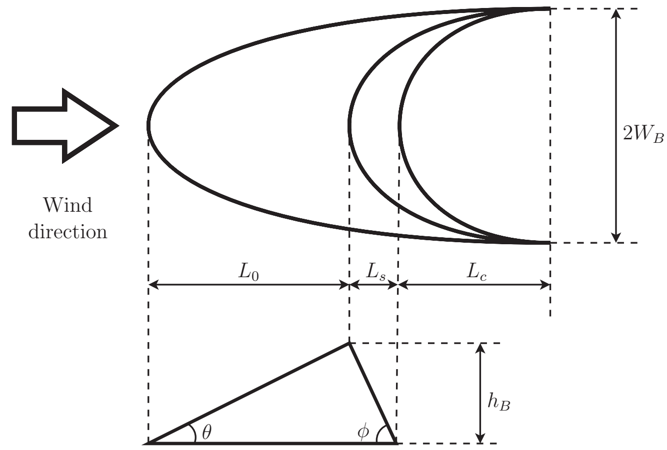

FIG. 6. Sketch of an ideal symmetrical barchan dune.

with

$$\kappa_\xi = \frac{q(BD_u + CD_d)}{4D_d D_u h_B}, \tag{A3}$$

$$\kappa_\zeta = \frac{Aq(D_d - D_u)}{4D_d D_u h_B}. \tag{A4}$$

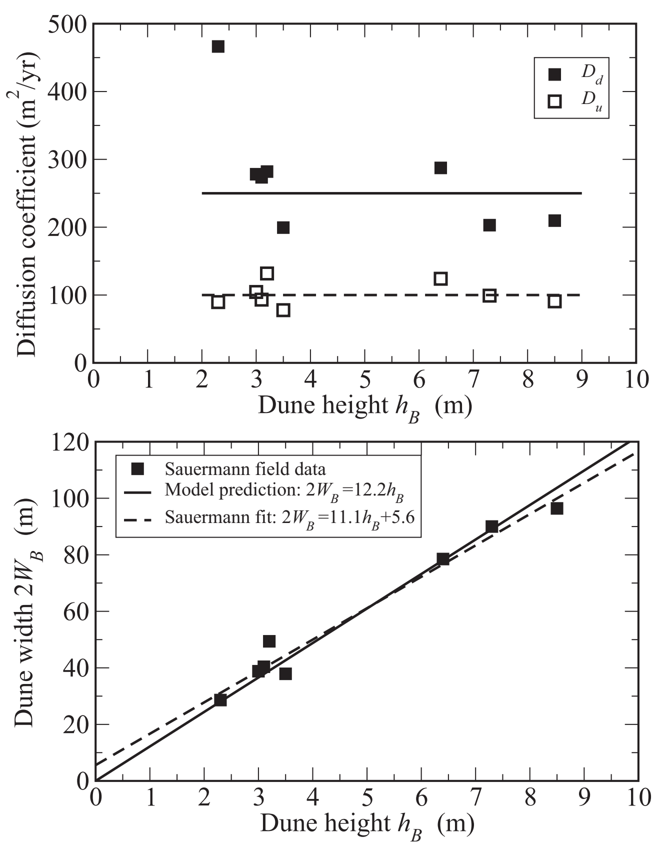

FIG. 7. (a) Diffusion coefficients D_u and D_d (on the upwind and downwind sides, respectively, of the dune) deduced from the field data reported in Ref. [8]: The solid and dashed lines represent the average values for D_d and D_u, respectively. (b) Dune width $2W_B$ versus the dune height h_B: comparison between symbols: the field data from Sauermann *et al.* [8] and solid line: the model prediction with $D_u = 100$, $D_d = 250$, and $q = 192$ m^2/yr.

Using Eqs. (A3) and (A4), we can derive an expression for D_u and D_d as a function of κ_ξ and κ_ζ and the geometrical parameters A, B, and C,

$$D_u = \frac{Aq}{4h_B}\frac{1}{A\kappa_\xi + B\kappa_\zeta}, \quad D_d = \frac{Aq}{4h_B}\frac{1}{A\kappa_\xi - C\kappa_\zeta}. \tag{A5}$$

The curvatures of κ_ξ and κ_ζ can simply be expressed as a function of the geomorphological parameters defined in Fig. 6,

$$\kappa_\xi = \frac{L_c + L_s}{W_B^2}, \quad \kappa_\zeta = \frac{h_B}{W_B^2}, \tag{A6}$$

as well as the geometrical parameters A, B, and C,

$$A = \frac{h_B}{L_0 + L_s}, \quad B = \frac{L_0}{L_0 + L_s}, \quad C = \frac{L_s}{L_0 + L_s}. \tag{A7}$$

Finally, plugging Eqs. (A6) and (A7) into Eqs. (A5) and taking advantage of the fact that $V_B = 3q/2h_B$ [cf. Eq. (18)], we get

$$D_u = \frac{V_B}{6}\frac{W_B^2}{L_c + L_0 + L_s}, \quad D_d = \frac{V_B}{6}\frac{W_B^2}{L_c}. \tag{A8}$$

Equations (A8) provide an expression of the diffusion coefficients D_u and D_d as a function of the geomorphological parameters of the barchan dune and its migration speed, which can easily be extracted from field measurements.

Using the field data of Sauermann *et al.* in Morocco [8], we can calculate the corresponding values of the diffusion coefficients for each barchan of the dune site [see Fig. 7(a)]. The diffusion coefficients D_u and D_d are found to be independent of the dune height (except for the smallest dune, which exhibits an abnormally high diffusion coefficient D_d) and are on the order of $D_u \approx 100$ and $D_d \approx 250$ m^2/yr, respectively. We can also estimate the sand flux q using the migration dune speed obtained from field measurements together with Eq. (13): $q = 192$ m^2/yr. Taking the above values for D_d, D_d, and q, from Eq. (18), we get a correlation between the dune width and its height ($2W_B = 12.2h_B$), which agrees well with the field data [see Fig. 7(b)].

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