Consistency and feasibility of approximate deconvolution models of turbulence

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CONSISTENCY AND FEASIBILITY OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

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Abstract. We prove that that the time averaged consistency error of the Nth approximate deconvolution LES model converges to zero uniformly in the kinematic viscosity and in the Reynolds number as the cube root of the averaging radius. We also give a higher order but non-uniform consistency error bound for the zeroth order model directly from the Navier-Stokes equations.

Key words. large eddy simulation, approximate deconvolution model, turbulence

1. Introduction. Direct numerical simulation of turbulent flows of incompressible, viscous fluids is often not computationally economical or even feasible. Thus, various turbulence models are used for simulations seeking to predict flow statistics or averages. In LES (large eddy simulation) the evolution of local, spatial averages is sought. Broadly, there are two types of LES models of turbulence: *descriptive* or phenomenological models (e.g., eddy viscosity models) and *predictive* models (considered herein). The accuracy of a model (meaning $||averagedNSEsolution - LESsolution||$) can be assessed in several experimental and analytical ways. One important approach (for which there are currently few results) is to study analytically the model's *consistency error* (defined precisely below) as a function of the averaging radius δ and the Reynolds number Re. The inherent difficulties are that (i) consistency error bounds for infinitely smooth functions hardy address essential features of turbulent flows such as irregularity and richness of scales, and (ii) worst case bounds for general weak solutions of the Navier Stokes equations are so pessimistic as to yield little insight. However, it is known that after time or ensemble averaging, turbulent velocity fields are often observed to have intermediate regularity as predicted by the Kolmogorov theory (often called the K41 theory), see, for example, [F95],[BIL04],[P00], [S01] , [Les97]. This case is often referred to as homogeneous isotropic turbulence and various norms of flow quantities can be estimated in this case using the K41 theory, Plancherel's Theorem and spectral integration. We mentioned Lilly's famous paper [L67] as an early and important example.

In this report we consider this third way begun in [LL04b]: consistency error bounds are developed for *time averaged*, fully developed, homogeneous, isotropic turbulence. Such bounds are inherently interesting and they also help answer two important related questions of *accuracy* and *feasibility* of LES. How small must δ be with respect to Re to have the average consistency error $<< O(1)$? Can consistency error $<< O(1)$ be attained for the cutoff length-scale δ within the inertial range?

Let the velocity $u(x,t) = u_j(x_1, x_2, x_3, t)$, $(j = 1, 2, 3)$ and pressure $p(x,t) = p(x_1, x_2, x_3, t)$ be a weak solution to the underlying Navier Stokes equations (NSE for short)

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f, \ and \ \nabla \cdot u = 0, \ in \ \mathbb{R}^3 \times (0, T), \tag{1.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, f is the body force, p is the pressure, and $\mathbb{R}^3$ is the flow domain. The above Navier-Stokes equations are supplemented by the

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initial condition, the usual pressure normalization condition

$$u(x,0) = u_0(x), \ and \int\limits_{\mathbb{R}^3} p dx = 0, \tag{1.2}$$

and appropriate boundary conditions. Our estimates are for the pure Cauchy problem; the role of boundary conditions at infinity is played by the assumption that the solution and all data are square integrable

$$for\ all\ t > 0 : \int\limits_{\mathbb{R}^3} |u(x,t)|^2 dx < \infty, \tag{1.3}$$

$$and \int\limits_{\mathbb{R}^3} |u_0(x)|^2 dx < \infty, \ and \int\limits_{\mathbb{R}^3} |f(x,t)|^2 dx < \infty, \ for\ 0 \le t. \tag{1.4}$$

We study a model for spacial averages of the fluid velocity with the following differential filter. Let δ denote the averaging radius; given $\phi \in L^2(\mathbb{R}^3)$, its average, denoted $\overline{\phi}$, is the solution in $H^1(\mathbb{R}^3)$ of the following problem [1]:

$$A\overline{\phi} := -(\frac{\delta}{L})^2 \triangle \overline{\phi} + \overline{\phi} = \phi. \tag{1.5}$$

The precise scaling in the above with respect to L is important in, for example, geophysical flow problems, [Lew97].Averaging the NSE shows that the true flow averages satisfy the (non-closed) equations

$$\overline{u}_t + \nabla \cdot (\overline{u\, u}) - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f}, \ and\ \nabla \cdot \overline{u} = 0. \tag{1.6}$$

The zeroth order model arises from $u \simeq \overline{u} + O(\delta^2)$, giving $\overline{u\, u} \simeq \overline{\overline{u}\, \overline{u}} + O(\delta^2)$. Calling w, q the resulting approximations to $\overline{u}, \overline{p}$, we obtain the model studied in [LL03],[LL04]:

$$w_t + \nabla \cdot (\overline{w\, w}) - \nu \triangle w + \nabla q = \overline{f}, \ and\ \nabla \cdot w = 0. \tag{1.7}$$

This zeroth order model's consistency error τ_0 is given by:

$$\tau_0 := \overline{u}\, \overline{u} - u\, u. \tag{1.8}$$

Subtracting the model from the averaged NSE, it is easy to see that the model's error, $\overline{u} - w$, satisfies $e(0,x) = 0, \nabla \cdot e = 0$ and

$$(\overline{u} - w)_t + \nabla \cdot (\overline{u\, u} - \overline{w\, w}) - \nu \triangle (\overline{u} - w) + \nabla (\overline{p} - q) = \nabla \cdot \overline{\tau_0} \tag{1.9}$$

which is driven only by the model's consistency error τ_0 through the term $\nabla \cdot \overline{\tau_0}$. Analysis of the dynamics of this error equation in [LL03], [LL04] showed that the modeling error is actually driven by τ_0 rather than $\nabla \cdot \overline{\tau_0}$. Since the model is stable and stable to perturbations, [LL04] , the *accuracy* of the model is governed by the size of various norms of its consistency error tensor τ_0.

The above example is the simplest (hence zeroth order) model in many families of LES models. We consider herein a family of Approximate Deconvolution Models (or ADM's) whose use in LES was pioneered by Stolz and Adams in a series of

[1] This precise definition of the differential filter is important since we consider dimensional scaling and L. It is obtained by rescaling x to x/L.

papers,[AS01], [SA99]. The size of the Nth models consistency error tensor directly determines the model's accuracy for these higher order model's as well, [DE04]. Let G_N $(N = 0, 1, 2, ...)$ denote the van Cittert, [BB98], approximate deconvolution operator which satisfies

$$u = G_N \overline{u} + O(\delta^{2N+2}), \text{ for smooth } u. \tag{1.10}$$

The models studied by Adams and Stolz are given by

$$w_t + \nabla \cdot (\overline{G_N w \, G_N w}) - \nu \triangle w + \nabla q + w' = \overline{f}, \ and\ \nabla \cdot w = 0. \tag{1.11}$$

The w'term is included to damp strongly the temporal growth of the fluctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on. Herein, we drop the w' term[2], select the averaging operator to be the above differential filter and (following Adams and Stolz) choose G_N to be the van Cittert approximation, [BB98],

$$G_N \phi := \sum_{n=0}^{N} (I - A^{-1})^n \phi. \tag{1.12}$$

For example, the induced closure model's corresponding to $N = 0$ and 1 are

$$G_0 \overline{u} = \overline{u}, so\ \overline{u\ u} \simeq \overline{\overline{u}\ \overline{u}} + O(\delta^2), \tag{1.13}$$

$$G_1 \overline{u} = 2\overline{u} - \overline{\overline{u}}, so\ \overline{u\ u} \simeq \overline{(2\overline{u} - \overline{\overline{u}})\ (2\overline{u} - \overline{\overline{u}})} + O(\delta^4). \tag{1.14}$$

To present the results, let $< \cdot >$ denote time averaging (defined precisely in section 2), δ the averaging radius used in the LES model, L a global length scale of the flow, Re the Reynolds number and U a characteristic velocity of the flow. For the generality we consider a good candidate for U to non-dimensionalize the equations is $U = time\ average\ of \{ \frac{1}{L^3} ||u(x,t)||^2_{L^2(\mathbb{R}^3)} \}^{\frac{1}{2}}$. Our estimates are based on three assumptions (whose plausibility is discussed in section 2). The first is consistent with this choice of U and reasonable for other choices as well.

Assumption 1. The equations are non-dimensionalized by a selection of U and L consistent with

$$< \frac{1}{L^3} ||u(x,t)||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} \leq U.$$

Assumption 2. The time averaged energy dissipation rate $\varepsilon(u)$, defined precisely in section 2, satisfies

$$\varepsilon(u) \leq C_1 \frac{U^3}{L}.$$

Assumption 3. The energy spectrum of the flow, defined precisely in section 2, satisfies

$$E(k) \leq \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

[2] The consistency error induced by adding the $w\prime$ term is smaller than that of the nonlinear term. While it does affect the model's dynamics, it does not affect the overall consistency error estimate.

1.1. The Zeroth Order Model. Consider first the case of the zeroth order model, (1.7) above. For the case $N = 0$ and for smooth u, it is easy to show that $\tau_0 = O((\frac{\delta}{L})^2)$.Indeed, simple estimates give $||u - \overline{u}||_{L^2(\mathbb{R}^3)} \leq (\frac{\delta}{L})^2 ||\triangle u||_{L^2(\mathbb{R}^3)}$, and thus since $\tau = \overline{u}\ (\overline{u} - u) + (\overline{u} - u)u$, it follows immediately that $||\tau_0||_{L^1(\mathbb{R}^3)} \leq 2||u||_{L^2(\mathbb{R}^3)} (\frac{\delta}{L})^2 ||\triangle u||_{L^2(\mathbb{R}^3)}$. In Section 3 we show that under K41 formalism the time average of this yields the bound

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq C_1^{\frac{3}{2}} \alpha^{\frac{1}{2}} \frac{U^2}{L^{\frac{1}{2}}} \mathrm{Re}^{\frac{5}{4}} (\frac{\delta}{L})^2 \tag{1.15}$$

While relevant in smooth regions of transitional flows, this smoothness, $\triangle u \in L^2(\mathbb{R}^3)$, needed does not describe the typical case of turbulent flows. Next we show in Section 3 that

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} \ > \leq 2C_1^{\frac{3}{2}}\ \mathrm{Re}^{\frac{1}{2}}\ L^2 U^4 \frac{\delta}{L}. \tag{1.16}$$

These two estimates, (1.15) and (1.16), are not sufficiently sharp to draw useful conclusions at higher Reynolds numbers (see Section 4). For example, this estimate suggests the zeroth order model is $O(\frac{\delta}{L})$ accurate only for $\frac{\delta}{L} << \mathrm{Re}^{-\frac{1}{2}}$. In our third estimate, using the K-41 phenomenology and spectral integration, we show, remarkably, the time averaged modeling consistency error is $O((\frac{\delta}{L})^{\frac{1}{3}})$ *uniformly in the Reynolds number Re and the kinematic viscosity* ν:

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq C_1^{\frac{1}{3}} \sqrt{\frac{72}{5}} U^2 L^{\frac{7}{6}} \varepsilon^{\frac{1}{3}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{1.17}$$

To illustrate the improvement of (1.17) over (1.16), suppressing all parameters except $\frac{\delta}{L}$ and Re, (1.16), and (1.17) together imply

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \simeq C \min\{(\frac{\delta}{L})^{\frac{1}{3}}, \mathrm{Re}^{\frac{1}{2}} (\frac{\delta}{L})\}. \tag{1.18}$$

The crossover point when (1.16) becomes sharper than (1.17) in (1.18) is when $(\frac{\delta}{L})^{\frac{1}{3}} \simeq \mathrm{Re}^{\frac{1}{2}} (\frac{\delta}{L})$, or equivalently $(\frac{\delta}{L}) \simeq \mathrm{Re}^{-\frac{3}{4}}$, i.e., only when the flow is fully resolved according to the classical estimates of the numbers of degrees of freedom in a turbulent flow!

1.2. The General Approximate Deconvolution Model. Section 3 gives consistency error estimates for the general case as well. The pointwise error in deconvolution by G_N can be calculated via the Neumann lemma following the approach in Lemma 2.3 in Dunca and Epshteyn [DE04]. The result, which we prove in section 3[3], is

$$u - G_N \overline{u} = (-1)^{N+1} (\frac{\delta}{L})^{2N+2} \triangle^{N+1} A^{-(N+1)} u. \tag{1.19}$$

As in the case of the zeroth order model, the model's error, $\overline{u} - w$, is driven by the term $\nabla \cdot \overline{\tau_N}$. Analysis of the dynamics of the model's error, $\overline{u} - w$, of the higher

[3] There is a minor error in Lemma 2.3 of [DE04] which affects the final result. For this reason, in Lemma 3.1 we give the corrected result and its proof.

order models in [DE04] shows that it is actually driven by its consistency error, τ_N, rather than $\nabla \cdot \overline{\tau_N}$, where τ_N is defined by

$$\tau_N := G_N \overline{u} G_N \overline{u} - u\ u \tag{1.20}$$

Adapting the ideas in the zeroth order case and using this last formula, in section 3 we give a first estimate of the model's consistency error τ_N

$$< ||\tau_N||_{L^1(\mathbb{R}^3)} > \leq \frac{2\alpha^{\frac{1}{2}}}{(2N+\frac{4}{5})^{\frac{1}{2}}} \frac{U^2}{L^{N-\frac{1}{2}}} (\frac{\delta}{L})^{2N+2} \operatorname{Re}^{\frac{3}{4}N+\frac{1}{2}}. \tag{1.21}$$

A second sharper estimate is then proven; remarkably, this estimate is uniform in the Reynolds number and, after computations and cancellations, takes the same form as in the zeroth order model above:

$$< ||\tau_N||_{L^1(\mathbb{R}^3)} > \leq (N+2)(3+\frac{2}{4N+\frac{10}{3}})^{\frac{1}{2}} \alpha^{\frac{1}{2}} U^2 L^{\frac{7}{6}} (\frac{\delta}{L})^{\frac{1}{3}}.$$

The impact of these estimates on practical issues in LES is considered in section 4.

2. The K-41 formalism. The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, ε, and the distribution of the flows kinetic energy across wave numbers, $E(k)$. Let $< \cdot >$ denote long time averaging

$$< \phi > (x) := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \phi(x,t) dt. \tag{2.1}$$

Time averaging is the original approach to turbulence of Reynolds, [R95]. It satisfies the following Cauchy-Schwartz inequality

$$< (\phi,\psi)_{L^2(\mathbb{R}^3)} > \quad \leq \quad < ||\phi||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} < ||\psi||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}}. \tag{2.2}$$

This follows, for example, by applying the usual Cauchy-Schwartz inequality on $\mathbb{R}^3 \times (0,T)$ followed by taking limits or from the connection with the inner product on the space of Besicovitch almost periodic functions, e.g., [Z85],[L84], [CB89].

Given the velocity field of a particular flow, $u(x,t)$, the (time averaged) energy dissipation rate of that flow is defined to be

$$\varepsilon := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \frac{1}{L^3} \int_{\mathbb{R}^3} \nu |\nabla u(x,t)|^2 dx\ dt. \tag{2.3}$$

It is known for many turbulent flows that the energy dissipation rate ε scales like $\frac{U^3}{L}$. This estimate, which is exactly Assumption 2, follows for homogeneous, isotropic turbulence from the K41 formalism, [F95], [Les97, Lesieur's book], [P00] and has been proven as an upper bound directly from the Navier Stokes equations for turbulent flows in bounded domains driven by persistent shearing of a moving boundary (rather than a body force), [CD92] ,[W97] . If $\widehat{u}(\mathbf{k},t)$ denotes the Fourier transform of $u(x,t)$ where $\mathbf{k}$ is the wave-number vector and $k = |\mathbf{k}|$ is its magnitude, then Plancherel's Theorem implies that the kinetic energy in u can be evaluated in physical space or in wave number space using the Fourier transform $\widehat{u}$ of u

$$\frac{1}{2}||u||^2_{L^2(\mathbb{R}^3)} = \frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx = \frac{1}{2}\int_{\mathbf{R}^3} |\widehat{u}(\mathbf{k},t)|^2 d\mathbf{k}. \tag{2.4}$$

Time averaging and rewriting the last integral in spherical coordinates gives

$$< \frac{1}{2}||u||^2_{L^2(\mathbb{R}^3)} >= \int_0^\infty E(k)dk, \quad where\ E(k) := \int_{|\mathbf{k}|=k} \frac{1}{2}|\widehat{<u>}(\mathbf{k},t)|^2 d\sigma. \tag{2.5}$$

The case of homogeneous, isotropic turbulence includes the assumption that (after time or ensemble averaging) $\widehat{u}(\mathbf{k})$ depends only on k and thus not the angles θ or φ. Thus, in this case,

$$E(k) = 2\pi k^2 |\widehat{<u>}(k)|^2. \tag{2.6}$$

Further, the K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\min} := U\nu^{-1} \leq k \leq \varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}} =: k_{\max} < \infty, \tag{2.7}$$

known as the inertial range, beyond which the kinetic energy in u is negligible, and in this range

$$E(k) \doteq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{2.8}$$

where $\alpha(\simeq 1.4)$ is the universal Kolmogorov constant, k is the wave number and ε is the particular flow's energy dissipation rate. The energy dissipation rate ε is the only parameter which differs from one flow to another. Outside the inertial range the kinetic energy in the small scales decays exponentially. Thus, we still have $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ since, after time averaging the energy in those scales is negligable, $E(k) \simeq 0$ for $k \geq k_{\max}$ and $E(k) \leq E(k_{\min})$ for $k \leq k_{\min}$. The fundamental assumption underlying our consistency error estimates is Assumption 3 that over all wave numbers

$$E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}.$$

Indeed, in figure 6.14 page 235 of [P00] the power spectrums of 17 different turbulent flows are plotted and the above bound is obvious in the plot.

3. Estimation of the consistency error. First note in all cases, the consistency error depends upon estimates of $u - G_N\overline{u}$ because

$$\tau_N = G_N\overline{u}\ G_N\overline{u} - u\ u = (G_N\overline{u} - u)G_N\overline{u} + u(G_N\overline{u} - u), N = 0, 1, 2, \cdots. \tag{3.1}$$

Consider τ_N. By the time averaged Cauchy-Schwartz inequality, and stability bounds for G_N we have

$$< ||\tau_N||_{L^1(\mathbb{R}^3)} > \leq (1 + ||G_N||) < ||u||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} < ||u - G_N\overline{u}||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} \tag{3.2}$$

Thus, estimates for the consistency error in $L^1(\mathbb{R}^3)$ flow from the above estimates of $||u - G_N\overline{u}||_{L^2(\mathbb{R}^3)}$ and later estimates of $< ||u - G_N\overline{u}||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}}$.

LEMMA 3.1. *For any* $\phi \in L^2$,

$$\phi - G_N\overline{\phi} = (I - A^{-1})^{N+1}\phi = (-1)^{N+1}(\frac{\delta}{L})^{2N+2}\triangle^{N+1}A^{-(N+1)}\phi.$$

Proof. Let $B = I - A^{-1}$. *Since* $\overline{\phi} = A^{-1}\phi, \overline{\phi} = (I - B)\phi$.*Since* $G_N := \sum_{n=0}^N B^n$,*a geometric series calculation gives* $(I - B)G_N\overline{\phi} = (I - B^{N+1})\overline{\phi}$. *Subtraction gives* $\phi - G_N\phi = AB^{N+1}\overline{\phi} = B^{N+1}A\overline{\phi} = B^{N+1}\phi$. *Finally,* $B = I - A^{-1}$, *so rearranging terms gives* $\phi - G_N\phi = (A - I)^{N+1}A^{-(N+1)}\phi = A^{-(N+1)}((-1)^{N+1}(\frac{\delta}{L})^{2N+2}\triangle^{N+1})\phi$,*which are the claimed results.* ∎

3.1. Estimates for the Zeroth Order Model. By multiplying (1.5) by $\overline{\phi}$, integrating by parts over $\mathbb{R}^3$ and using the Cauchy-Schwartz inequality on the right hand side, it follows readily that the averaging process is stable and smoothing in the sense

$$||\overline{\phi}||,\ 2\frac{\delta}{L}||\nabla\overline{\phi}||,\ \text{and}\ \frac{1}{2}(\frac{\delta}{L})^2||\triangle\overline{\phi}||\ \leq ||\phi||. \tag{3.3}$$

Denote the averaging error by $\Phi = (\phi - \overline{\phi})$.Using the equation $-(\frac{\delta}{L})^2\triangle\Phi + \Phi = -(\frac{\delta}{L})\triangle\phi$, the following error bounds for Φ follow in much the same ways as the above stability bounds

$$||\phi - \overline{\phi}|| \leq \frac{1}{\sqrt{2}}(\frac{\delta}{L})||\nabla\phi||,\ ||\nabla(\phi - \overline{\phi})|| \leq \frac{1}{\sqrt{2}}(\frac{\delta}{L})||\triangle\phi||,\ and\ ||\phi - \overline{\phi}|| \leq (\frac{\delta}{L})^2||\triangle\phi||. \tag{3.4}$$

Detailed proofs of such estimates in these and other norms are given in [LL04].

Consider τ. By the time-averaged Cauchy-Schwartz inequality (2.2) and the above stability bounds we have

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} >\leq 2 < ||u||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} < ||u - \overline{u}||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} \tag{3.5}$$

Estimates for τ thus follow from estimates for $||u - \overline{u}||_{L^2(\mathbb{R}^3)}$ and $< ||u - \overline{u}||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}}$.

It is possible to get a very quick estimate of $< ||\tau_0||_{L^1(\mathbb{R}^3)} >$ by scaling, simple inequalities and Assumptions 1 through 3 as follows. Holder's inequality and the above simple estimate $||u - \overline{u}||_{L^2(\mathbb{R}^3)} \leq (\frac{\delta}{L})||\nabla u||$ give

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} >\leq 2L^3U(\frac{\delta}{L})\nu^{-\frac{1}{2}} < \frac{1}{L^3}\nu||\nabla u||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}}\leq 2\nu^{-\frac{1}{2}}L^3U(\frac{\delta}{L})\varepsilon^{\frac{1}{2}}. \tag{3.6}$$

Assumption 2 is that the energy dissipation rate ε scales like $C_1\frac{U^3}{L}$. As noted in section2, this assumption is consistent with the K-41 formalism, [F95], [Les97, Lesieur's book] . The estimate $\varepsilon \leq C_1\frac{U^3}{L}$ has also been proven directly from the Navier-Stokes equations for turbulent shear flows in bounded domains by [CD92] ,[W97] . Using this upper bound for ε gives the bound (1.16)

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} >\leq C_1^{\frac{1}{2}}L^2U^{\frac{7}{2}}\,\mathrm{Re}^{\frac{1}{2}}\ \frac{\delta}{L}. \tag{3.7}$$

This bounds implies the model becomes accurate already in the inertial range as certain flow features begin to resolve. However, it is also pessimistic since it requires $\frac{\delta}{L} << \mathrm{Re}^{-\frac{1}{2}}$ for accuracy, see section 4. This estimate comes directly from the Navier Stokes equations and is thus independent of the K41 theory. Remarkably, using K41 it is improvable to one uniform in the Reynolds number in the case of homogeneous, isotropic turbulence.

The related estimate (1.15) is obtained by using instead

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} >\leq 2 < ||u||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}} < ||u - \overline{u}||^2_{L^2(\mathbb{R}^3)} >^{\frac{1}{2}}\leq 2UL^{\frac{3}{2}}(\frac{\delta}{L})^2 < ||\triangle u||^2 >^{\frac{1}{2}}\ . \tag{3.8}$$

The term $< ||\triangle u||^2 >^{\frac{1}{2}}$can be estimated in the case of homogeneous, isotropic turbulence using spectral integration as follows

$$< ||\triangle u||^2 >= \int_{k_0}^{k_{\max}} k^4E(k)dk \leq \alpha\varepsilon^{\frac{2}{3}}\int_0^{k_{\max}} k^{\frac{7}{3}}dk = .3\alpha\varepsilon^{\frac{2}{3}}(\varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}})^{\frac{10}{3}}. \tag{3.9}$$

Using the estimate $\varepsilon \leq C_1 \frac{U^3}{L}$ and rearranging the resulting RHS into terms involving the Reynolds number gives

$$< ||\triangle u||^2 >^{\frac{1}{2}} \leq C_1^{\frac{3}{4}} \alpha^{\frac{1}{2}} \frac{U}{L^2} \operatorname{Re}^{\frac{5}{4}} (\frac{\delta}{L})^2, \tag{3.10}$$

which gives

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq C_1^{\frac{3}{4}} \alpha^{\frac{1}{2}} \frac{U^2}{L^{\frac{1}{2}}} \operatorname{Re}^{\frac{5}{4}} (\frac{\delta}{L})^2. \tag{3.11}$$

This is an asymptotically higher power of $\frac{\delta}{L}$ for moderate Reynolds numbers but it yields the consistency condition $\frac{\delta}{L} << \operatorname{Re}^{-\frac{5}{8}}$ which is worse than the preceding one.

The sharper bound (1.17) is proven as follows. Under the K41 formalism we can write

$$< ||u - \overline{u}||^2_{L^2(\mathbb{R}^3)} > \leq \int_{k_0}^{k_{\max}} (1 - \frac{1}{(\frac{\delta}{L})^2 k^2 + 1})^2 E(k) dk, \tag{3.12}$$

where $(0 <) k_0 (\leq k_{min})$ is the smallest frequency, and $k_{max} (= \varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}})$ the largest frequency. Over the inertial range $E(k) = \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ and outside it $E(k) \leq \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$. Thus, we can write

$$< ||u - \overline{u}||^2_{L^2(\mathbb{R}^3)} > \leq 2\alpha \varepsilon^{\frac{2}{3}} \int_{k_0}^{k_{\max}} (\frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1})^2 k^{-\frac{5}{3}} dk =: I. \tag{3.13}$$

The remainder of the work in (1.17) is direct estimation of the above integral. The integral I requires different treatments for small and large wave numbers. We shall thus estimate the two cases separately depending on which term in the denominator ($(\frac{\delta}{L})^2 k^2$ or 1) is dominant. The transition point is the cutoff wave number $\frac{L}{\delta}$; thus we break it into two integrals at this point

$$I := I_{low} + I_{high}, \text{where } I_{low} = \int_{k_0}^{\frac{L}{\delta}} ...dk, \text{and } I_{high} = \int_{\frac{L}{\delta}}^{k_{high}} ...dk. \tag{3.14}$$

For the low frequency components we have

$$\frac{1}{2} (\frac{\delta}{L})^2 k^2 \leq \frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1} \leq (\frac{\delta}{L})^2 k^2, \text{for } 0 \leq k \leq \frac{L}{\delta}, \tag{3.15}$$

and thus

$$I_{low} \leq 2(\frac{\delta}{L})^4 \alpha \varepsilon^{\frac{2}{3}} \int_{k_0}^{\frac{L}{\delta}} k^{\frac{7}{3}} dk \leq 2(\frac{\delta}{L})^4 \alpha \varepsilon^{\frac{2}{3}} \int_0^{\frac{L}{\delta}} k^{\frac{7}{3}} dk = \frac{3}{5} \alpha \varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.16}$$

Consider the second integral; over the high frequency components

$$I_{high} = 2\alpha \varepsilon^{\frac{2}{3}} \int_{\frac{L}{\delta}}^{k_{\max}} (\frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1})^2 k^{-\frac{5}{3}} dk. \tag{3.17}$$

For the high frequency components we have

$$\frac{1}{2} \leq \frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1} \leq 1, \text{for } \frac{L}{\delta} \leq k \leq \infty. \tag{3.18}$$

Using this estimate, the integral becomes

$$I_{high} \leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{L}{\delta}}^{k_{\max}} k^{-\frac{5}{3}} dk \leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{L}{\delta}}^{\infty} k^{-\frac{5}{3}} dk. \tag{3.19}$$

This leads to the upper estimate for I_{high} :

$$I_{high} \leq 3\alpha\varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.20}$$

Adding these estimates we obtain

$$I \leq \frac{18}{5}\alpha\varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.21}$$

Using this bound for I in $< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq 2UL^{\frac{3}{2}} I^{\frac{1}{2}}$ gives the following estimate for the model's consistency error

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq \sqrt{\frac{72}{5}} UL^{\frac{3}{2}} \varepsilon^{\frac{1}{3}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{3.22}$$

Using the estimate for $\varepsilon \leq C_1 \frac{U^3}{L}$, (independent of Re and ν), we obtain the claimed estimate (1.17)

$$< ||\tau_0||_{L^1(\mathbb{R}^3)} > \leq C_1^{\frac{1}{3}} \sqrt{\frac{72}{5}} U^2 L^{\frac{7}{6}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{3.23}$$

This is remarkable in that it predicts the models consistency error to approach zero uniformly in the Reynolds number.

3.2. The General Approximate Deconvolution Model. The analysis in the case $N = 1, 2, 3, \cdots$ follows the zeroth order case using stability of G_N and the estimates

$$< ||\tau_N||_{L^1} > \leq (1 + ||G_N||) < ||u||_{L^2}^2 >^{\frac{1}{2}} < ||u - G_N\overline{u}||_{L^2}^2 >^{\frac{1}{2}} . \tag{3.24}$$

Indeed, beginning with $< ||u||_{L^2}^2 >^{\frac{1}{2}} \leq UL^{\frac{3}{2}}$, we have

$$< ||\tau_N||_{L^1(\mathbb{R}^3)} > \leq (1 + ||G_N||) UL^{\frac{3}{2}} < ||u - G_N\overline{u}||_{L^2(\mathbb{R}^3)}^2 >^{\frac{1}{2}} . \tag{3.25}$$

First, note that by the spectral mapping theorem the operator norm $||G_N||$ is easily bounded

$$||G_N|| = \sum_{n=0}^{N} \lambda_{\max}(I - A^{-1})^n = \sum_{n=0}^{N} (I - \frac{1}{\lambda_{\max}})^n = N + 1. \tag{3.26}$$

As in subsection 3.1, we use spectral integration to evaluate the deconvolution approximation's consistency error. Lemma 3.1 implies

$$I := < ||u - G_N\overline{u}||_{L^2(\mathbb{R}^3)}^2 > = 2 \int_{k_0}^{k_{\max}} (\frac{(\frac{\delta}{L})^2 k^2}{1 + (\frac{\delta}{L})^2 k^2})^{2N+2} E(k) dk. \tag{3.27}$$

Since $E(k) \leq \alpha\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ we have

$$I \leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{k_0}^{k_{\max}} (\frac{(\frac{\delta}{L})^2 k^2}{1 + (\frac{\delta}{L})^2 k^2})^{2N+2} k^{-\frac{5}{3}} dk.$$

The integral I has different asymptotics for low and high wave-numbers. As in the zeroth order case, the transition depends upon which term in the denominator is dominant. Thus, split

$$I := I_{low} + I_{high}, \text{where } I_{low} = \int_{k_0}^{\frac{L}{\delta}} ...dk, \text{and } I_{high} = \int_{\frac{L}{\delta}}^{k_{high}} ...dk. \tag{3.28}$$

For the low frequencies we have

$$I_{low} \le (\frac{\delta}{L})^{4N+4} 2\alpha\varepsilon^{\frac{2}{3}} \int_0^{\frac{L}{\delta}} k^{4N+\frac{7}{3}} dk = \frac{2\alpha\varepsilon^{\frac{2}{3}}}{4N+\frac{10}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.29}$$

Thus,

$$I_{low} \le \frac{2\alpha\varepsilon^{\frac{2}{3}}}{4N+\frac{10}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.30}$$

For the high wave numbers the dominant term in the denominator is the k^2 term. We thus have

$$I_{high} \le 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{L}{\delta}}^{k_{\max}} k^{-\frac{5}{3}} dk \le 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{L}{\delta}}^{\infty} k^{-\frac{5}{3}} dk \le 3\alpha\varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}, \tag{3.31}$$

or, collecting these two estimates,

$$I \le (3 + \frac{2}{4N+\frac{10}{3}}) \alpha\varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.32}$$

Using the bound $\varepsilon \le C_1 \frac{U^3}{L}$ gives the sharper (and longer) estimate

$$< ||\tau_N||_{L^1(\mathbb{R}^3)} > \le C_1^{\frac{1}{3}} (N+2)(3 + \frac{2}{4N+\frac{10}{3}})^{\frac{1}{2}} \alpha^{\frac{1}{2}} U^2 L^{\frac{7}{6}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{3.33}$$

4. Conclusion: Feasibility of LES. For LES with deconvolution models to be feasible for fully developed turbulence two competing restrictions on the averaging radius must simultaneously be satisfied. First, $\frac{\delta}{L}$ must be well inside the inertial range, giving a lower bound on the averaging radius, $\frac{\delta}{L} >> \varepsilon^{-\frac{1}{4}}\nu^{\frac{3}{4}}$. Second, the models consistency error must be small: $< ||\tau|| ><< 1$. We have seem that this gives an upper bound on $\frac{\delta}{L}$which decreases as Re increases. For LES to be useful, these two constraints must be satisfied simultaneously.

To illustrate the competition between these two constraints, consider the zeroth order model first and suppress all constants except $\frac{\delta}{L}$ and Re. Using the consistency error bound (1.16) yields a narrow band of possible values of the averaging radius

$$C\,\mathrm{Re}^{-\frac{3}{4}} << \frac{\delta}{L} << C\,\mathrm{Re}^{-\frac{1}{2}}. \tag{4.1}$$

Thus, the extra analysis required is important for giving an accurate analytical assessment of LES. Indeed, using instead the sharper estimate $< ||\tau_0|| > \le C(\frac{\delta}{L})^{\frac{1}{3}}$, predicts feasibility of any of the approximate deconvolution models provided

$$C\,\mathrm{Re}^{-\frac{3}{4}} << \frac{\delta}{L} << O(1). \tag{4.2}$$

In most applications, turbulent flows simulations require a (more) universal model which is accuracy for the application's heterogeneous mix of laminar and transitional regions, boundary layers and fully developed turbulence. The entire family of approximate deconvolution models shares a remarkable global accuracy for fully developed turbulence. On the other hand, the higher order models are significantly more accurate in the laminar and transitional regions. The overall analytic conclusion is that higher order models are preferable to lower order models up to the point where their computational cost become prohibitive. This observation, while surprising from the point of view of traditional error analysis, is consistent with the extensive experiments in the work of Stolz and Adams with the models.

Our results raise (at least) four questions worthy of further study. At this point we do not know if the global accuracy of $O((\frac{\delta}{L})^{\frac{1}{3}})$ is an essential feature of fully developed turbulence or is possibly improvable as models advance. Second, the global consistency error is interesting but the distribution of those errors among scales might be essential. Third, a broader understanding of an LES model's consistency errors requires developing estimates directly from the Navier-Stokes equations- a very hard analytical problem. Fourth, understanding how an LES model's dynamics redistributes consistency errors into modeling errors is also a critical question for LES. Finally, we note that the results herein are extendable from the Cauchy problem using Fourier transforms (herein) to L-periodic problems using Fourier series.

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